

$AdS_3 \times S^3 \times M^4$ string S-matrices from unitarity cuts

Lorenzo Bianchi and Ben Hoare

*Institut für Physik, Humboldt-Universität zu Berlin
Newtonstraße 15, 12489 Berlin, Germany*

{lorenzo.bianchi, ben.hoare}@physik.hu-berlin.de

Abstract

Continuing the program initiated in arXiv:1304.1798 we investigate unitarity methods applied to two-dimensional integrable field theories. The one-loop computation is generalized to encompass theories with different masses in the asymptotic spectrum and external leg corrections. Additionally, the prescription for working with potentially singular cuts is modified to cope with an ambiguity that was not encountered before. The resulting methods are then applied to three light-cone gauge string theories; i) $AdS_3 \times S^3 \times T^4$ supported by RR flux, ii) $AdS_3 \times S^3 \times S^3 \times S^1$ supported by RR flux and iii) $AdS_3 \times S^3 \times T^4$ supported by a mix of RR and NSNS fluxes. In the first case we find agreement with the exact result following from symmetry considerations and in the second case with one-loop semiclassical computations. This agreement crucially includes the rational terms and hence supports the conjecture that S-matrices of integrable field theories are cut-constructible, up to a possible shift in the coupling. In the final case, under the assumption that our methods continue to give all rational terms, we give a conjecture for the one-loop phases.

Contents

1	Introduction	3
2	General principles	5
2.1	Theories with a single mass	5
2.2	Theories with multiple masses	8
2.3	External leg corrections	10
2.4	Structure of the result	12
3	Massive sector for $AdS_3 \times S^3 \times M^4$ supported RR flux	14
3.1	Exact and tree-level S-matrices	14
3.1.1	Massive sector for $AdS_3 \times S^3 \times T^4$	14
3.1.2	Massive sector for $AdS_3 \times S^3 \times S^3 \times S^1$	15
3.1.3	A general tree-level S-matrix for the $AdS_3 \times S^3 \times M^4$ theories	18
3.2	Result from unitarity techniques	19
3.2.1	Coefficients of the logarithms	19
3.2.2	Rational terms from the s-channel	20
3.2.3	The t-channel contribution and the dressing phases	21
3.2.4	External leg corrections for $AdS_3 \times S^3 \times S^3 \times S^1$	22
4	Massive sector for $AdS_3 \times S^3 \times T^4$ supported by mixed flux	25
4.1	Exact S-matrix and tree-level result	25
4.2	Result from unitarity techniques	27
5	Comments	28
A	Notation and conventions	29
A.1	$AdS_3 \times S^3 \times M^4$ supported by RR flux	29
A.2	$AdS_3 \times S^3 \times T^4$ supported by mixed flux	31
B	Phase factors for $AdS_3 \times S^3 \times M^4$ backgrounds	32

1 Introduction

In [1, 2] unitarity methods [3, 4, 5] were applied to various two-dimensional field theories with the aim of computing the two-particle to two-particle S-matrix. One of the intriguing consequences of these works was the observation that this approach is particularly powerful when applied to integrable field theories.¹ In [1] it was observed that the one-loop S-matrices, including rational terms, for a number of integrable theories (including sine-Gordon and various generalizations) are completely cut-constructible (up to possible finite shifts in the coupling). In particular, the unitarity construction automatically accounts for additional contributions from one-loop counterterms that are required to preserve integrability in the standard Feynman diagram computation. Furthermore, as the unitarity construction reduces the one-loop computation to scalar bubble integrals, which are finite in two dimensions, issues of regularization are bypassed.

The motivation for investigating these methods in the context of two-dimensional integrable field theories is their potential use in the study of string backgrounds with worldsheet integrability. The classic example is the $AdS_5 \times S^5$ superstring [8], whose worldsheet theory was demonstrated to be classically integrable in [9]. In [1] it was shown that the one-loop unitarity computation of the light-cone gauge S-matrix, the tree-level components of which were computed in [10], matches the expansion of the exact S-matrix [11, 12], including rational terms. The exact S-matrix was derived as a result of an extensive body of work, following from considerations of symmetries, integrability and perturbation theory in both gauge and string theory. For further details and references see the reviews [13, 14]. In [2] the one- and two-loop logarithmic terms were also shown to be in agreement. In general, in an integrable theory the logarithmic terms are required to exponentiate into a set of overall phase factors. This was confirmed to be the case for a variety of integrable string backgrounds of relevance in the context of the AdS/CFT correspondence [2, 15].

In this paper we will develop and apply the methods of [1] to a class of integrable theories that arise as the light-cone gauge-fixing [16] of $AdS_3 \times S^3 \times M^4$ string backgrounds [17, 18, 19, 20, 21]. We will focus on the following three cases. The first is the simplest and is when the compact manifold is T^4 with the background supported by RR flux. The second is when the compact manifold is $S^3 \times S^1$, again supported by RR flux. For the last we return to T^4 , but with the background now supported by a mix of RR and NSNS fluxes. The final two can both be thought of (at least at the level of the decompactified light-cone gauge worldsheet theory) as deformations of the first. For the $S^3 \times S^1$ theory we have a parameter $\alpha \in [0, 1]$, such that for $\alpha = 0$ and 1, one or other of the two three-spheres blows up, while in the mixed flux theory we have a parameter $q \in [0, 1]$, where $q = 0$ corresponds to pure RR flux and $q = 1$ to pure NSNS flux.

The S-matrix of interest to us describes the scattering of excitations on the decompactified string worldsheet in the uniform light-cone gauge [16]. The masses of the asymptotic excitations are given by the expansion around the BMN string [22]. For the theories under consideration we have the following spectra

¹S-matrices of integrable field theories are in general rather special as they are heavily constrained by a set of physical requirements – no particle production, equality of the sets of incoming and outgoing momenta and the factorization of the n -particle scattering amplitude into a product of two-particle S-matrices [6, 7].

Theory	Spectrum	
$AdS_3 \times S^3 \times T^4$ (RR flux)	$(4+4) \times 1$	$(4+4) \times 0$
$AdS_3 \times S^3 \times S^3 \times S^1$ (RR flux)	$(2+2) \times 1$	$(2+2) \times \alpha$
	$(2+2) \times 1 - \alpha$	$(2+2) \times 0$
$AdS_3 \times S^3 \times T^4$ (mixed flux)	$(4+4) \times \sqrt{1-q^2}$	$(4+4) \times 0$

where $(n+n)$ denotes bosons+fermions. As expected, in each case we have $(8+8)$ excitations in total and the masses of the bosons match those of the fermions. All three cases feature massless modes, which need careful treatment in two dimensions. In the main text we will argue that if we restrict to massive external legs, then we can ignore the massless modes completely in the one-loop unitarity computation. Therefore in this paper we will do so, leaving the question of massless modes in two-dimensional unitarity computations for future work.

The main input of the one-loop unitarity computation is the tree-level S-matrix of the theory. Various components of the tree-level S-matrices for the T^4 and $S^3 \times S^1$ backgrounds supported by RR flux were computed in [23, 24], and in [25] for the mixed flux case. These results, along with the symmetries and integrability of the theory, can be used to completely determine the tree-level S-matrix. For the $AdS_3 \times S^3 \times S^3 \times S^1$ background we will also require additional input. In this case the expansion of the light-cone gauge-fixed Lagrangian contains odd powers of the fields, in particular cubic terms [26, 27]. This, along with the asymptotic mass spectrum, implies that there can be external leg corrections contributing to the one-loop unitarity computation. We deal with these additional terms by considering a unitarity computation involving form factors (i.e. partially off-shell quantities) on either side of the cut.

In [2] the one- and two-loop logarithmic terms were studied for these theories. Therefore, our main aim here is to compute the rational terms. The motivation for this is two-fold. First, we would like to demonstrate that the matrix structure (that is the S-matrix up to overall phase factors) matches that which is found via symmetries and integrability [28, 29, 30, 31, 32, 33]. Second, we would like to investigate the phases of these theories, including the rational terms. Thus far there is only an all-loop conjecture (supported by semiclassical one-loop computations in [34, 35]²) for the phases in the $AdS_3 \times S^3 \times T^4$ case supported by RR flux [31]. There is also a semiclassical one-loop computation of the phases in the $AdS_3 \times S^3 \times S^3 \times S^1$ case in [36]. In these two models we check that our methods reproduces the known results at one-loop, providing further support for the conjecture of [1] that the S-matrices of integrable theories are cut-constructible (up to possible shifts in the coupling). Given this, we can therefore use unitarity techniques to conjecture one-loop expressions for the phases in the mixed flux case, which should then provide an insight into how to deform the all-loop phases of [31].

We shall start in section 2 with an outline of the general method, emphasizing the modifications to the construction in [1] that are required to deal with the three theories of interest. There are three such modifications – a prescription for the rational t-channel contribution when the consistency condition of [1] is not satisfied, a discussion of what happens with particles of different mass in the asymptotic spectrum and the possibility of external leg corrections. In the following sections the procedure is then applied to the $AdS_3 \times S^3 \times M^4$ string backgrounds.

²The two semiclassical one-loop computations [34, 35] are not in complete agreement. While the logarithmic terms match, the rational terms we find from unitarity methods agree with those in [34] and the expansion of the exact result [31]. It is currently unclear what the precise reason for the disagreement between [34] and [35] is.

2 General principles

In [1] unitary methods were used to construct the one-loop S-matrix of various two-dimensional integrable field theories. Agreement with known results was found not only for the logarithmic terms, but also with the rational part (up to shifts in the coupling). The construction of the rational part was based on a prescription that dealt with various singular cuts that arise when applying the usual unitarity methods. This prescription required that a consistency condition on the tree-level S-matrix was satisfied. However, for various other examples of interest in the context of integrable string backgrounds, this consistency condition is not satisfied. In this section we will outline the construction of [1] (see also [2] for a discussion of the logarithmic terms to two loops) and describe a modification of the prescription that will allow the method to be used when the consistency condition is not satisfied.

2.1 Theories with a single mass

The object of interest is the two-particle S-matrix, defined in terms of the four-point amplitude

$$\langle \Phi^P(q) \Phi^Q(q') | \mathbb{S} | \Phi_M(p) \Phi_N(p') \rangle = \mathcal{A}_{MN}^{PQ}(p, p', q, q') . \quad (2.1)$$

Here \mathbb{S} is the scattering operator, M, N, \dots are indices running over the particle content of the theory and p, p', q, q' are the on-shell momenta of the fields. For now we will restrict to the case where all the particles have equal non-vanishing mass, which we set to one. As a consequence of momentum conservation, the four-point amplitude takes the form

$$\mathcal{A}_{MN}^{PQ}(p, p', q, q') = (2\pi)^2 \delta^{(2)}(p + p' - q - q') \tilde{\mathcal{A}}_{MN}^{PQ}(p, p', q, q') . \quad (2.2)$$

Furthermore, at four points, two-dimensional kinematics implies that the set of initial momenta is preserved in the scattering process, as demonstrated by the following identity

$$\delta^{(2)}(p + p' - q - q') = \frac{\mathcal{J}(p, p')}{4\epsilon\epsilon'} (2\epsilon \delta(p - q) 2\epsilon' \delta(p' - q') + 2\epsilon \delta(p - q') 2\epsilon' \delta(p' - q)) , \quad (2.3)$$

where p, p', q, q' are the spatial momentum and the Jacobian $\mathcal{J}(p, p') = 1/(\partial\epsilon/\partial p - \partial\epsilon'/\partial p')$ depends on the on-shell energies $\epsilon(p), \epsilon'(p')$. Note that we have assumed the particle velocities are ordered as $v = \partial\epsilon/\partial p > \partial\epsilon'/\partial p' = v'$ and for the spatial momentum δ -functions we have used a normalization that becomes the standard Lorentz-invariant one in the relativistic case.

Substituting (2.3) into (2.2) we find two terms. Without loss of generality we can consider just the amplitude associated to the first product of δ -functions, $2\epsilon \delta(p - q) 2\epsilon' \delta(p' - q')$. The two-particle S-matrix is then defined as

$$S_{MN}^{PQ}(p, p') \equiv \frac{\mathcal{J}(p, p')}{4\epsilon\epsilon'} \tilde{\mathcal{A}}_{MN}^{PQ}(p, p', p, p') , \quad (2.4)$$

As usual we can expand in powers of the coupling

$$S_{MN}^{PQ}(p, p') = \delta_M^P \delta_N^Q + i\hbar^{-1} T^{(0)PQ}_{MN}(p, p') + i\hbar^{-2} T^{(1)PQ}_{MN}(p, p') + \mathcal{O}(\hbar^{-3}) , \quad (2.5)$$

with the identity at leading order, followed by the tree-level S-matrix $T^{(0)}$ at order \hbar^{-1} and the n -loop S-matrix $T^{(n)}$ at order \hbar^{-n-1} .

In this paper we will be interested in computing the cut-constructible part of $T^{(1)}$ from the tree-level S-matrix $T^{(0)}$ for the light-cone gauge S-matrices for string theories in $AdS_3 \times S^3 \times M^4$ backgrounds. For these theories the inverse string tension plays the role of the coupling \hbar^{-1} . It is well known that the

following tensor contractions

$$(A \circledast B)_{MN}^{PQ}(p, p') = A_{MN}^{RS}(p, p') B_{RS}^{PQ}(p, p') , \quad (2.8)$$

$$(A \circledcirc B)_{MN}^{PQ}(p, p') = (-1)^{([P]+[S])([Q]+[R])} A_{MR}^{SQ}(p, p') B_{SN}^{PR}(p, p') , \quad (2.9)$$

$$(A \circledcircleft B)_{MN}^{PQ}(p, p') = (-1)^{[P][S]+[R][S]} A_{MR}^{SP}(p, p') B_{SN}^{RQ}(p, p') , \quad (2.10)$$

$$(A \circledcircright B)_{MN}^{PQ}(p, p') = (-1)^{[Q][R]+[R][S]} A_{MR}^{PS}(p, p') B_{SN}^{QR}(p, p') , \quad (2.11)$$

where $[M] = 0$ for a boson and 1 for a fermion, and the following scalar bubble integrals

$$I_s \equiv I((p+p')^2, 1, 1) = \frac{1}{4(e'p - ep')} \left(1 - \frac{\text{arcsinh}(e'p - ep')}{i\pi} \right) = \frac{J}{i\pi} (i\pi - \theta) , \quad (2.12)$$

$$I_t \equiv I(0, 1, 1) = \frac{1}{4\pi i} , \quad (2.13)$$

$$I_u \equiv I((p-p')^2, 1, 1) = \frac{1}{4(e'p - ep')} \frac{\text{arcsinh}(e'p - ep')}{i\pi} = \frac{J\theta}{i\pi} , \quad (2.14)$$

where we have used (2.6) and defined

$$\theta \equiv \text{arcsinh}(e'p - ep') . \quad (2.15)$$

The final step is to set $q = p$ and $q' = p'$ to extract the one-loop S-matrix

$$T^{(1)} = \frac{iJ}{2} (C_s I_s + C_t I_t + C_u I_u) , \quad (2.16)$$

where for clarity we have suppressed the indices. Here J is the contribution from the Jacobian and the $\frac{1}{2}$ is the symmetry factor. The matrices $C_{s,u}$ are given by

$$C_s = \tilde{T}^{(0)} \circledast \tilde{T}^{(0)} , \quad C_u = \tilde{T}^{(0)} \circledcircright \tilde{T}^{(0)} , \quad (2.17)$$

where

$$\tilde{T}^{(0)} = J^{-1} T^{(0)} . \quad (2.18)$$

The t-channel contraction is more subtle as there two possible choices for freezing the loop momenta (i.e. in terms of p and q or p' and q') giving potentially different results. These correspond to the two contractions in eqs. (2.10) and (2.11). In the theories considered in [1] it turned out that these two choices always gave the same result, i.e.

$$\tilde{T}^{(0)} \circledcircleft \tilde{T}^{(0)} = \tilde{T}^{(0)} \circledcircright \tilde{T}^{(0)} . \quad (2.19)$$

and this issue could safely be ignored. However, for the S-matrices we will consider in this paper, i.e. the light-cone gauge S-matrices for strings in $AdS_3 \times S^3 \times M^4$ this is no longer the case. In particular we note that the function $\tilde{T}^{(0)}(p, p)$ cannot have any momentum dependence in a relativistic theory,³ whereas in a non-relativistic theory it can depend on p , generating an asymmetry between p and p' .⁴ Hence it is natural to conjecture that we should take the average of the two contractions. Therefore

$$C_t = \frac{1}{2} (\tilde{T}^{(0)} \circledcircleft \tilde{T}^{(0)} + \tilde{T}^{(0)} \circledcircright \tilde{T}^{(0)}) . \quad (2.20)$$

³Let us recall that in a relativistic theory the S-matrix depends only on the difference of rapidities, which vanishes for $p' = p$.

⁴In [1] the only non-relativistic theory that was considered was the light-cone gauge-fixed string in $AdS_5 \times S^5$ for which, despite the asymmetry, eq. (2.19) still holds.

To conclude the construction we can use the explicit expressions of the integrals $I_{s,t,u}$ in eqs. (2.12) to (2.14) and the relation between $T^{(0)}$ and $\tilde{T}^{(0)}$ (2.18) to rewrite the one-loop result as

$$T^{(1)} = \frac{\theta}{2\pi}(T^{(0)} \circledast T^{(0)} - T^{(0)} \circledast T^{(0)}) + \frac{i}{2}T^{(0)} \circledast T^{(0)} + \frac{1}{16\pi}(\tilde{T}^{(0)} \circledast T^{(0)} + T^{(0)} \circledast \tilde{T}^{(0)}) , \quad (2.21)$$

where, under the assumption that $T^{(0)}$ is real, there is a natural split of the result into three pieces; a logarithmic part, an imaginary rational part, and a real rational part.

2.2 Theories with multiple masses

We will now generalize the above construction to the case where the asymptotic spectrum contains particles of different mass. In this derivation we will restrict to theories whose tree-level S-matrix is integrable, in particular, using the consequence that the set of outgoing momenta is a permutation of the set of incoming momenta. This means that, for the reasons explained in section 2.1, tadpoles and one-loop graphs built from a three- and five-point amplitude will be ignored in the unitarity computation. Therefore we are again left with the three contributions given in figure 1.

We consider the configuration in which the external legs with indices M and P have mass m and the associated momenta are equal ($p = q$) and N and Q have mass m' with $p' = q'$.⁵ For the s- and u-channels the story is then largely the same as the single-mass case. It follows from the assumptions outlined in the previous paragraph that when the two propagators are cut the internal loop momenta are frozen to the values of the external momenta. The tree-level amplitudes on either side of the cut can then be pulled out of the integral and we are left with scalar bubble integrals with coefficients given by contractions of tree-level amplitudes. Working through the remaining steps, which are essentially identical to the single-mass case, it is clear that the contribution from these graphs is given by

$$T_{s,u}^{(1)} = \frac{\theta}{2\pi}(T^{(0)} \circledast T^{(0)} - T^{(0)} \circledast T^{(0)}) + \frac{i}{2}T^{(0)} \circledast T^{(0)} , \quad (2.22)$$

where

$$\begin{aligned} \theta &\equiv \operatorname{arcsinh}\left(\frac{e'p - ep'}{mm'}\right) , & e &= \sqrt{p^2 + m^2} , & e' &= \sqrt{p'^2 + m'^2} . \\ I_s &\equiv I((p + p')^2, m, m') = \frac{1}{4(e'p - ep')} \left(1 - \frac{\operatorname{arcsinh}(\frac{e'p - ep'}{mm'})}{i\pi}\right) = \frac{J}{i\pi}(i\pi - \theta) , \\ I_u &\equiv I((p - p')^2, m, m') = \frac{1}{4(e'p - ep')} \frac{\operatorname{arcsinh}(\frac{e'p - ep'}{mm'})}{i\pi} = \frac{J\theta}{i\pi} , \end{aligned} \quad (2.23)$$

Here m and m' are the masses of the two particles being scattered and the scalar bubble integral $I(P^2, m, m')$ is defined in eq. (2.7). Eq. (2.22) therefore fixes the logarithmic and imaginary rational parts of the one-loop result.

The real rational part, which comes from the t-channel contribution, is, as before, more subtle. In the single-mass case, the guiding principle for computing the t-channel cuts was to only fix $q = p$ and $q' = p'$ at the end in order to avoid ill-defined expressions in the intermediate steps. Therefore, let us consider the t-channel graph in figure 1 with the external legs with indices M and P having mass m , N and Q mass m' and the loop legs mass m_l , but p, q, p' and q' kept arbitrary, i.e. we do *not* fix $q = p$ and $q' = p'$.

⁵Our procedure implies that if we assume the set of outgoing momenta is equal to a permutation of the set of incoming momenta at tree level, this property automatically extends to one loop.

After putting the loop legs on-shell the loop momenta are fixed by the momentum conservation delta functions in terms of the external momenta. Solving in terms of p and q we find

$$\begin{aligned} l_{1\pm}^\uparrow &= \frac{1}{2} [q_\pm - p_\pm + \sqrt{(q_\pm - p_\pm)^2 + 4\frac{m_l^2}{m^2} q_\pm p_\pm}] , \\ l_{2\pm}^\uparrow &= \frac{1}{2} [p_\pm - q_\pm + \sqrt{(p_\pm - q_\pm)^2 + 4\frac{m_l^2}{m^2} p_\pm q_\pm}] , \end{aligned} \quad (2.24)$$

while solving in terms of p' and q' gives

$$\begin{aligned} l_{1\pm}^\downarrow &= \frac{1}{2} [p'_\pm - q'_\pm + \sqrt{(p'_\pm - q'_\pm)^2 + 4\frac{m_l^2}{m^2} p'_\pm q'_\pm}] , \\ l_{2\pm}^\downarrow &= \frac{1}{2} [q'_\pm - p'_\pm + \sqrt{(q'_\pm - p'_\pm)^2 + 4\frac{m_l^2}{m'^2} q'_\pm p'_\pm}] , \end{aligned} \quad (2.25)$$

where the light-cone momenta are defined in appendix A. The first solution (2.24) then gives a contribution proportional to

$$(-1)^{[P][S]+[R][S]} \tilde{\mathcal{A}}_{MR}^{(0)SP}(p, l_1^\uparrow, l_2^\uparrow, q) \tilde{\mathcal{A}}_{SN}^{(0)RQ}(l_2^\uparrow, p', l_1^\uparrow, q') . \quad (2.26)$$

The arguments of the second factor of $\tilde{\mathcal{A}}^{(0)}$ contain all four of the external momenta and therefore this part is well-defined when we fix $q = p$ and $q' = p'$. Therefore, let us focus on the first factor of $\tilde{\mathcal{A}}^{(0)}$, whose arguments only depend on two of the momenta. Recalling that in an integrable theory the amplitude should vanish unless the set of outgoing momenta is a permutation of the set of incoming momenta, it follows that this first factor vanishes unless $m_l = m$. In this case (2.26) reduces to

$$(-1)^{[P][S]+[R][S]} \tilde{\mathcal{A}}_{MR}^{(0)SP}(p, q, p, q) \tilde{\mathcal{A}}_{SN}^{(0)RQ}(p, p', q, q') . \quad (2.27)$$

Finally setting $q = p$ and $q' = p'$ this expression can then be written in terms of tree-level S-matrices. A similar logic follows for the second solution (2.25), except that here the contribution vanishes unless $m_l = m'$.

It therefore follows that the contribution from the t-channel is given by

$$T_t^{(1)} = \frac{1}{16\pi} \left(\frac{1}{m^2} \tilde{T}^{(0)} \overset{\text{t}}{\leftarrow} T^{(0)} + \frac{1}{m'^2} T^{(0)} \overset{\text{t}}{\rightarrow} \tilde{T}^{(0)} \right) , \quad (2.28)$$

where $\tilde{T}^{(0)}$ in the first term is built from the tree-level S-matrix for the scattering of two excitations of mass m , while in the second term it is built from the tree-level S-matrix for two excitations of mass m' . We have included an additional factor of $\frac{1}{2}$ as we should still use both vertices to solve for the loop momenta and take the average.

Combining eqs. (2.22) and (2.28) we find that the one-loop result in the case where an excitation of mass m is scattered with an excitation of mass m' is given by

$$T^{(1)} = \frac{\theta}{2\pi} (T^{(0)} \overset{\text{u}}{\circ} T^{(0)} - T^{(0)} \overset{\text{s}}{\circ} T^{(0)}) + \frac{i}{2} T^{(0)} \overset{\text{s}}{\circ} T^{(0)} + \frac{1}{16\pi} \left(\frac{1}{m^2} \tilde{T}^{(0)} \overset{\text{t}}{\leftarrow} T^{(0)} + \frac{1}{m'^2} T^{(0)} \overset{\text{t}}{\rightarrow} \tilde{T}^{(0)} \right) , \quad (2.29)$$

where, again under the assumption that $T^{(0)}$ is real, there is a natural split of the result into three pieces; a logarithmic part, an imaginary rational part, and a real rational part. Setting $m = m' = 1$ we see that this formula reduces to, and hence incorporates, the single-mass case given in eq. (2.21).

A key consequence of the results in this section is that the cut-constructible one-loop S-matrix for the scattering of a particle of mass m with one of mass m' is built from the corresponding tree-level S-matrix along with the tree-level S-matrices for the scattering of two particles of mass m and for two particles of

mass m' , both evaluated at equal momenta. In particular there are no contributions containing tree-level S-matrices for particles of masses other than m and m' . This will be important in later sections as it allows us to construct the one-loop cut-constructible S-matrix for various sectors without knowing the full tree-level S-matrix.

The result (2.29) deserves a comment regarding its relation to integrability and the Yang-Baxter equation. The Yang-Baxter equation is a cubic matrix equation that should be satisfied by S-matrices describing scattering in integrable theories. Up to signs related to fermions, which we are not concerned with for this schematic discussion, it can be written as

$$\mathbb{S}_{12}\mathbb{S}_{13}\mathbb{S}_{23} = \mathbb{S}_{23}\mathbb{S}_{13}\mathbb{S}_{12} , \quad (2.30)$$

where these operators are acting on a three-particle state and the indices denote the particles that are being scattered. The first non-trivial order in its perturbative expansion is called the classical Yang-Baxter equation and is a relation that is quadratic in the tree-level S-matrix,

$$[\mathbb{T}_{12}^{(0)}, \mathbb{T}_{13}^{(0)}] + [\mathbb{T}_{12}^{(0)}, \mathbb{T}_{23}^{(0)}] + [\mathbb{T}_{13}^{(0)}, \mathbb{T}_{23}^{(0)}] = 0 . \quad (2.31)$$

At the next order we find the following relation

$$\begin{aligned} [\mathbb{T}_{12}^{(0)}, \mathbb{T}_{13}^{(1)}] + [\mathbb{T}_{12}^{(0)}, \mathbb{T}_{23}^{(1)}] + [\mathbb{T}_{13}^{(0)}, \mathbb{T}_{23}^{(1)}] - [\mathbb{T}_{13}^{(0)}, \mathbb{T}_{12}^{(1)}] - [\mathbb{T}_{23}^{(0)}, \mathbb{T}_{12}^{(1)}] - [\mathbb{T}_{23}^{(0)}, \mathbb{T}_{13}^{(1)}] = \\ \mathbb{T}_{23}^{(0)}\mathbb{T}_{13}^{(0)}\mathbb{T}_{12}^{(0)} - \mathbb{T}_{12}^{(0)}\mathbb{T}_{13}^{(0)}\mathbb{T}_{23}^{(0)} . \end{aligned} \quad (2.32)$$

One can check that, assuming that the tree-level S-matrix satisfies the classical Yang-Baxter equation (2.31), the rational s-channel contribution to the cut-constructible one-loop S-matrix precisely cancels the terms cubic in the tree-level S-matrix on the right-hand side of eq. (2.32). Therefore, for the one-loop cut-constructible S-matrix to respect integrability the remaining terms should satisfy (2.32) with the right-hand side set to zero. In general, this condition is not easy to solve, but two solutions are clear. The first is the tree-level S-matrix itself (which amounts to a shift in the coupling), and the second is any contribution that can be absorbed into the overall phase factors.

It will turn out that of the three theories we are interested in, two satisfy this property. For the $AdS_3 \times S^3 \times S^3 \times S^1$ background, the one-loop cut-constructible S-matrix as defined by (2.29) has a rational piece coming from the t-channel that does not satisfy (2.32) with zero on the right-hand side. However, there is a meaning to these terms – they are cancelled by corrections to the external legs, which we will now discuss.

2.3 External leg corrections

In the construction outlined thus far we have not included any discussion of corrections to the external legs. As shall become apparent, for the $AdS_3 \times S^3 \times S^3 \times S^1$ background, these will be important even at one loop. These corrections will give a rational contribution to the S-matrix and can follow from the three types of Feynman diagrams in figure 2.

We will be interested in external leg corrections at one loop that are caught by unitarity. In order to approach this problem let us first review how external leg corrections are usually dealt with in a standard Feynman diagram calculation. We denote the sum of all one particle irreducible insertions into a scalar propagator as $-i\Sigma(p) = -ih^{-1}\Sigma^{(1)}(p) + \mathcal{O}(h^{-2})$, where $-ih^{-1}\Sigma^{(1)}(p)$ is the one-loop contribution. After re-summing one finds

$$\text{---} \bigcirc \text{---} = \frac{i}{p^2 - m^2 - \Sigma(p)} \quad (2.33)$$

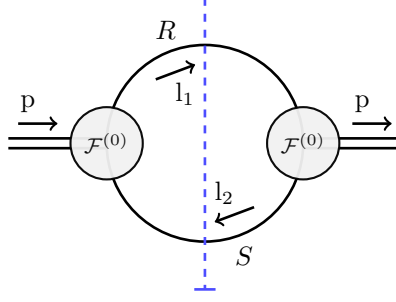


Figure 3: Cut of a two-point function obtained by fusing two form factors. The double line indicates an off-shell state.

Now taking figure 3 with a mass $m_1 - m_2$ external particle,⁷ internal particles with masses m_1 and m_2 corresponding to momentum l_1 and l_2 and returning p off-shell, the explicit expression for this diagram is given by

$$\Sigma^{(1)}(p)|_{cut} = \int \frac{d^2 l_1}{(2\pi)^2} i\pi \delta^+(l_1^2 - m_1^2) i\pi \delta^+((l_1 - p)^2 - m_2^2) \mathcal{F}_{RS}^{(0)}(p, l_1, l_1 - p) \mathcal{F}_{RS}^{(0)\dagger}(p, l_1, l_1 - p) . \quad (2.36)$$

Here, as in the unitarity computation of the S-matrix, the cut completely freezes the internal momenta:

$$l_1 = \frac{m_1^2 - m_2^2 + p^2 - \sqrt{\Delta}}{2p^2} p \equiv l_* , \quad (2.37)$$

$$p - l_1 = \frac{m_2^2 - m_1^2 + p^2 + \sqrt{\Delta}}{2p^2} p \equiv l'_* , \quad (2.38)$$

where $\Delta = p^4 + m_1^4 + m_2^4 - 2m_1^2 p^2 - 2m_2^2 p^2 - 2m_1^2 m_2^2$. It therefore follows that we can pull the numerators out of the integrand and uplift the integral as was done for the four-point amplitude. This gives

$$\Sigma^{(1)}(p) = \frac{1}{2} \left| \mathcal{F}_{RS}^{(0)}(p, l_*, l'_*) \right|^2 I(p^2, m_1, m_2) , \quad (2.39)$$

with the integral $I(p^2, m_1, m_2)$ defined in (2.7). In section 3.2.4 we will apply this formula to a specific example and we will also point out the limits of its application.

2.4 Structure of the result

To conclude this section let us make some remarks about the features of the result that are relevant for the integrable light-cone gauge S-matrices for strings in $AdS_3 \times S^3 \times M^4$ backgrounds. These theories all have the property that the massive excitations can be grouped into particles and antiparticles transforming with charge $\sigma = +1$ and $\sigma = -1$ under a global $U(1)$ symmetry. Furthermore, not only is the set of incoming momenta preserved by the scattering process, but so are the $U(1)$ charges associated to the individual momenta, i.e. $\sigma_M = \sigma_P$ and $\sigma_N = \sigma_Q$. The general structure of the S-matrix is then

$$S_{MN}^{PQ}(p, p') = \exp[i\varpi_{\sigma_M \sigma_N}(p, p')] \hat{S}_{MN}^{PQ}(p, p') , \quad (2.40)$$

⁷This will be the case we consider for $AdS_3 \times S^3 \times S^3 \times S^1$. One can also consider a mass m_1 external particle and internal particles with masses $m_1 - m_2$ and m_2 ($m_1 > m_2$). In this case the two loop momenta in figure 3 should be pointing in the same direction.

where ϖ are the phases and the matrix structure \hat{S} is fixed by the symmetry of the theory. Each of these objects admit a perturbative expansion at strong coupling (i.e. around $h = \infty$):

$$S = \mathbf{1} + i \sum_{n=1}^{\infty} h^{-n} T^{(n-1)}, \quad \hat{S} = \mathbf{1} + i \sum_{n=1}^{\infty} h^{-n} \hat{T}^{(n-1)}, \quad \varpi_{\sigma_M \sigma_N}(p, p') = \sum_{n=1}^{\infty} h^{-n} \varpi_{\sigma_M \sigma_N}^{(n-1)}(p, p'). \quad (2.41)$$

Furthermore, as \hat{S} is fixed by symmetries it should contain no logarithmic functions of the momenta. Therefore, all the logarithms are contained in the phases, and to the one-loop order we can separate these off as follows

$$\varpi_{\sigma_M \sigma_N}^{(0)}(p, p') = \phi_{\sigma_M \sigma_N}^{(0)}(p, p'), \quad \varpi_{\sigma_M \sigma_N}^{(1)}(p, p') = \ell_{\sigma_M \sigma_N}(p, p') \theta + \phi_{\sigma_M \sigma_N}^{(1)}(p, p'). \quad (2.42)$$

Here θ , defined in eq. (2.15), is the only possible logarithm appearing at one loop, and $\phi_{\sigma_M \sigma_N}^{(n)}$ are rational functions of the momenta.

Substituting eqs. (2.41) and (2.42) into (2.40) we find

$$T^{(0)} = \phi_{\sigma_M \sigma_N}^{(0)}(p, p') \mathbf{1} + \hat{T}^{(0)}, \quad (2.43)$$

$$T^{(1)} = \ell_{\sigma_M \sigma_N}(p, p') \theta \mathbf{1} + \frac{i}{2} \left[\phi_{\sigma_M \sigma_N}^{(0)}(p, p') \right]^2 \mathbf{1} + \phi_{\sigma_M \sigma_N}^{(1)}(p, p') \mathbf{1} + i \phi_{\sigma_M \sigma_N}^{(0)}(p, p') \hat{T}^{(0)} + \hat{T}^{(1)}. \quad (2.44)$$

Let us compare the structure of the one-loop result following from integrability (2.44) with that following from unitarity methods (2.21), (2.29). The comparison between the two expressions leads to the following identifications (note that by definition the functions $\ell_{\sigma_M \sigma_N}$ and $\phi_{\sigma_M \sigma_N}^{(n)}$ are real)

$$\frac{1}{2\pi} (T^{(0)} \circledast T^{(0)} - T^{(0)} \circledcirc T^{(0)}) = \ell_{\sigma_M \sigma_N}(p, p') \mathbf{1}, \quad (2.45)$$

$$\begin{aligned} \frac{1}{2} T^{(0)} \circledcirc T^{(0)} &= \frac{1}{2} \left[\phi_{\sigma_M \sigma_N}^{(0)}(p, p') \right]^2 \mathbf{1} + \phi_{\sigma_M \sigma_N}^{(0)}(p, p') \hat{T}^{(0)} + \text{Im}(\hat{T}^{(1)}) \\ &\Rightarrow \frac{1}{2} \hat{T}^{(0)} \circledcirc \hat{T}^{(0)} = \text{Im}(\hat{T}^{(1)}), \end{aligned} \quad (2.46)$$

$$\frac{1}{16\pi} \left(\frac{1}{m^2} \tilde{T}^{(0)} \circledast T^{(0)} + \frac{1}{m'^2} T^{(0)} \circledast \tilde{T}^{(0)} \right) + (\Sigma_1^{(1)}(p) + \Sigma_1^{(1)}(p')) T^{(0)} = \phi_{\sigma_M \sigma_N}^{(1)}(p, p') \mathbf{1} + \text{Re}(\hat{T}^{(1)}), \quad (2.47)$$

where we have assumed that $T^{(0)}$ is real, which will indeed be the case for all the models we consider. For the rational terms coming from the s-channel in (2.46) we have simplified the expression that needs to be checked by substituting in for $T^{(0)}$ (2.43) and using that $\mathbf{1} \circledcirc \mathbf{1} = \mathbf{1}$, $\hat{T}^{(0)} \circledcirc \mathbf{1} = \hat{T}^{(0)}$ and $\mathbf{1} \circledcirc \hat{T}^{(0)} = \hat{T}^{(0)}$ are satisfied by definition (see eq. (2.8)). In (2.47) we have also included a possible contribution from external leg corrections to the real rational part of $T^{(1)}$ (see eq. (2.35)), as discussed in section 2.3. Eqs. (2.45), (2.46) and (2.47) are therefore the three equations that we need to check to see how much of the exact S-matrix is recovered from the unitarity construction.

Factoring out an overall phase factor as in (2.40) clearly contains a degree of arbitrariness. Of course, this choice should not affect the final result, however, there are certain choices that interplay well with the unitarity construction. In particular, if there is a scattering process for which the only possible outgoing two-particle state is the incoming state ($M = P = M_*$, $N = Q = N_*$), then the corresponding amplitude must be a phase factor. In this case we can set

$$\hat{S}_{M_* N_*}^{M_* N_*} = 1, \quad (2.48)$$

where M_* and N_* are fixed and there is no sum. This choice is consistent with (2.46) – both sides are clearly vanishing by construction. Furthermore, $\phi^{(1)}$ is just given by the t-channel contraction (plus possible external leg corrections) with indices $M = P = M_*$, $N = Q = N_*$.

3 Massive sector for $AdS_3 \times S^3 \times M^4$ supported RR flux

In this section we apply the methods of section 2 to the massive sectors of the $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ light-cone gauge-fixed string theories supported by RR flux.

3.1 Exact and tree-level S-matrices

We start by reviewing some of the available results for the light-cone gauge S-matrices for the $AdS_3 \times S^3 \times M^4$ string theories supported by RR flux.

3.1.1 Massive sector for $AdS_3 \times S^3 \times T^4$

The quadratic light-cone gauge-fixed action for the $AdS_3 \times S^3 \times T^4$ background describes $4 + 4$ massive and $4 + 4$ massless fields. Here we will just consider the scattering of two massive excitations to two massive excitations. The S-matrix of the theory was fixed up to two phases in [30] using symmetries.

Thinking of the particle content of the massive sector as $2 + 2$ complex degrees of freedom, we label these fields as $\Phi_{\varphi\varphi}$, $\Phi_{\psi\psi}$, $\Phi_{\varphi\psi}$ and $\Phi_{\psi\varphi}$, and their complex conjugates as $\Phi_{\bar{\varphi}\bar{\varphi}}$, $\Phi_{\bar{\psi}\bar{\psi}}$, $\Phi_{\bar{\varphi}\bar{\psi}}$ and $\Phi_{\bar{\psi}\bar{\varphi}}$, where we understand φ , $\bar{\varphi}$ as bosonic and ψ , $\bar{\psi}$ as fermionic indices.

As a consequence of the symmetries and integrability of the theory, the S-matrix factorizes:

$$\mathbb{S} |\Phi_{M\dot{M}}(p)\Phi_{N\dot{N}}(p')\rangle = (-1)^{[\dot{M}][N]+[\dot{N}][Q]} S_{MN}^{PQ}(p, p') S_{\dot{M}\dot{N}}^{\dot{P}\dot{Q}}(p, p') |\Phi_{P\dot{P}}(p)\Phi_{Q\dot{Q}}(p')\rangle, \quad (3.1)$$

where the indices take the following values: $\{\varphi, \bar{\varphi}, \psi, \bar{\psi}\}$. One can check that the construction outlined in section 2 gives the same one-loop result whether we consider the factorized or full S-matrix. Therefore, for simplicity we will work with the former. The general structure of the factorized S-matrix takes the form given in (2.40) with $\sigma_{\varphi} = \sigma_{\psi} = +$ and $\sigma_{\bar{\varphi}} = \sigma_{\bar{\psi}} = -$. Charge conjugation symmetry implies that $\phi_{++} = \phi_{--}$, $\phi_{+-} = \phi_{-+}$, $\ell_{++} = \ell_{--}$ and $\ell_{+-} = \ell_{-+}$. Therefore, in the following we will focus on the $++$ and $+-$ sectors. A typical feature of the uniform light-cone gauge is the dependence of the phase on a gauge-fixing parameter a . This dependence has the following exact form

$$\exp \left[\frac{i}{2} (a - \frac{1}{2}) (\epsilon' p - \epsilon p') \right], \quad (3.2)$$

where the all-order energies ϵ are defined in appendix A.1.

As we discussed in section 2.4 we define the overall phase factors by setting particular components of \hat{S}_{MN}^{PQ} to one

$$\hat{S}_{\varphi\varphi}^{\varphi\varphi}(p, p') = 1, \quad \hat{S}_{\varphi\bar{\psi}}^{\varphi\bar{\psi}}(p, p') = 1. \quad (3.3)$$

The parametrizing functions of the exact S-matrix are defined as

$$\begin{aligned} S_{\varphi\varphi}^{\varphi\varphi}(p, p') &= A_{++}(p, p') & S_{\varphi\bar{\varphi}}^{\varphi\bar{\varphi}}(p, p') &= A_{+-}(p, p') \\ S_{\varphi\psi}^{\varphi\psi}(p, p') &= B_{++}(p, p') & S_{\varphi\bar{\psi}}^{\varphi\bar{\psi}}(p, p') &= B_{+-}(p, p') \\ S_{\varphi\bar{\psi}}^{\psi\bar{\psi}}(p, p') &= C_{++}(p, p') & S_{\varphi\bar{\psi}}^{\psi\bar{\psi}}(p, p') &= C_{+-}(p, p') \\ S_{\psi\varphi}^{\psi\varphi}(p, p') &= D_{++}(p, p') & S_{\psi\bar{\varphi}}^{\psi\bar{\varphi}}(p, p') &= D_{+-}(p, p') \\ S_{\psi\psi}^{\psi\psi}(p, p') &= E_{++}(p, p') & S_{\psi\bar{\psi}}^{\psi\bar{\psi}}(p, p') &= E_{+-}(p, p') \\ S_{\psi\bar{\psi}}^{\psi\bar{\psi}}(p, p') &= F_{++}(p, p') & S_{\psi\bar{\psi}}^{\psi\bar{\psi}}(p, p') &= F_{+-}(p, p') \end{aligned} \quad (3.4)$$

with the functions in string frame given by [30]

$$\begin{aligned}
A_{++}(p, p') &= S_{++}^{11}(p, p') , & B_{++}(p, p') &= S_{++}^{11}(p, p') \frac{x'^+ - x^+}{x'^+ - x^-} \frac{1}{\nu} , \\
C_{++}(p, p') &= S_{++}^{11}(p, p') \frac{x'^+ - x'^-}{x'^+ - x^-} \frac{\eta}{\eta'} \sqrt{\frac{\nu'}{\nu}} , & D_{++}(p, p') &= S_{++}^{11}(p, p') \frac{x'^- - x^-}{x'^+ - x^-} \nu' , \\
E_{++}(p, p') &= S_{++}^{11}(p, p') \frac{x^+ - x^-}{x'^+ - x^-} \frac{\eta'}{\eta} \sqrt{\frac{\nu'}{\nu}} , & F_{++}(p, p') &= S_{++}^{11}(p, p') \frac{x'^- - x^+}{x'^+ - x^-} \frac{\nu'}{\nu} , \\
A_{+-}(p, p') &= S_{+-}^{11}(p, p') \frac{1 - \frac{1}{x^+ x'^-}}{1 - \frac{1}{x^- x'^-}} \nu , & B_{+-}(p, p') &= -S_{+-}^{11}(p, p') \frac{i \eta \eta'}{x^- x'^-} \frac{1}{1 - \frac{1}{x^- x'^-}} (\nu \nu')^{-\frac{1}{2}} , \\
C_{+-}(p, p') &= S_{+-}^{11}(p, p') , & D_{+-}(p, p') &= S_{+-}^{11}(p, p') \frac{1 - \frac{1}{x^+ x'^+}}{1 - \frac{1}{x^- x'^-}} \nu \nu' , \\
E_{+-}(p, p') &= S_{+-}^{11}(p, p') \frac{1 - \frac{1}{x^- x'^+}}{1 - \frac{1}{x^- x'^-}} \nu' , & F_{+-}(p, p') &= -S_{+-}^{11}(p, p') \frac{i \eta \eta'}{x^+ x'^+} \frac{1}{1 - \frac{1}{x^- x'^-}} (\nu \nu')^{\frac{3}{2}} .
\end{aligned} \tag{3.5}$$

The definitions of the variables x^\pm entering these expressions are given for general mass in appendix A.1. Here the masses should be set to one. The functions $S_{++}^{11}(p, p')$ and $S_{+-}^{11}(p, p')$ are two overall phase factors, i.e. in the notation of eq. (2.42) $S_{\sigma_M \sigma_N}^{11}(p, p') = e^{i \varpi_{\sigma_M \sigma_N}^{11}(p, p')}$. The superscripts refer to the masses of the two particles being scattered. These phase factors are not fixed by symmetry. They are, however, constrained by crossing symmetry and a conjecture for their exact expressions was given in [31], supported by semiclassical one-loop computations [34, 35] (see footnote 2). More details are given in appendix B.

The input needed to apply the unitarity construction described in section 2 is the tree-level S-matrix. Various components were computed directly in [23, 24]. These are consistent with the near-BMN expansion of the exact result (3.5), (3.6), for which we recall that the integrable coupling used in the definition of x^\pm has the expansion $h(h) = h + \mathcal{O}(h^0)$ and to take the near-BMN limit the spatial momenta should first be rescaled; $p \rightarrow \frac{p}{h}$. The remaining components of the tree-level S-matrix can then be fixed from the expansion of the exact result. Here we shall present the result in the gauge $a = \frac{1}{2}$ as the dependence on a goes through the unitarity procedure without any particular subtlety, i.e. it exponentiates as in eq. (3.2). The tree-level S-matrix reads

$$\begin{aligned}
A_{++}^{(0)}(p, p') &= \frac{(p + p')^2}{4(e'p - ep')} , & B_{++}^{(0)}(p, p') &= \frac{p'^2 - p^2}{4(e'p - ep')} , \\
C_{++}^{(0)}(p, p') &= pp' \frac{\sqrt{(e + p)(e' - p')} + \sqrt{(e - p)(e' + p')}}{2(e'p - ep')} , & D_{++}^{(0)}(p, p') &= -\frac{p'^2 - p^2}{4(e'p - ep')} , \\
E_{++}^{(0)}(p, p') &= pp' \frac{\sqrt{(e + p)(e' - p')} + \sqrt{(e - p)(e' + p')}}{2(e'p - ep')} , & F_{++}^{(0)}(p, p') &= -\frac{(p + p')^2}{4(e'p - ep')} ,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
A_{+-}^{(0)}(p, p') &= \frac{(p - p')^2}{4(e'p - ep')} , & B_{+-}^{(0)}(p, p') &= pp' \frac{\sqrt{(e - p)(e' + p')} - \sqrt{(e + p)(e' - p')}}{2(e'p - ep')} , \\
C_{+-}^{(0)}(p, p') &= \frac{p'^2 - p^2}{4(e'p - ep')} , & D_{+-}^{(0)}(p, p') &= -\frac{p'^2 - p^2}{4(e'p - ep')} , \\
E_{+-}^{(0)}(p, p') &= -\frac{(p - p')^2}{4(e'p - ep')} , & F_{+-}^{(0)}(p, p') &= pp' \frac{\sqrt{(e - p)(e' + p')} - \sqrt{(e + p)(e' - p')}}{2(e'p - ep')} .
\end{aligned} \tag{3.8}$$

3.1.2 Massive sector for $AdS_3 \times S^3 \times S^3 \times S^1$

The quadratic light-cone gauge-fixed action for the $AdS_3 \times S^3 \times S^3 \times S^1$ background describes particles with four different masses. The field content is summarised in table 1. Here we will focus on the scattering

of massive states with masses α and $\bar{\alpha} = 1 - \alpha$.

Fields	Mass
$\varphi_1, \bar{\varphi}_1, \chi^1, \bar{\chi}^1$	$m_1 = 1$
$\varphi_2, \bar{\varphi}_2, \chi^2, \bar{\chi}^2$	$m_2 = \alpha$
$\varphi_3, \bar{\varphi}_3, \chi^3, \bar{\chi}^3$	$m_3 = \bar{\alpha}$
$\varphi_4, \bar{\varphi}_4, \chi^4, \bar{\chi}^4$	$m_4 = 0$

Table 1: Field content of the $AdS_3 \times S^3 \times S^3 \times S^1$ light-cone gauge-fixed string theory.

Let us first analyze the S-matrix for $AdS_3 \times S^3 \times S^3 \times S^1$ describing the scattering of two particles of mass α .⁸ When we restrict to this sector the S-matrix has the same structure as the factorized S-matrix for $AdS_3 \times S^3 \times T^4$, again taking the form given in (2.40). The tree-level S-matrix, however, is different and this will have non-trivial consequences for the unitarity calculation. Compared to the $AdS_3 \times S^3 \times T^4$ case the dependence on the gauge-fixing parameter a is modified due to the fact that this is now the full S-matrix. The new expression reads

$$\exp \left[i(a - \frac{1}{2})(\epsilon' p - \epsilon p') \right] . \quad (3.9)$$

We again use (3.3) to choose the overall phase factors and define the parametrizing functions as in eq. (3.4).⁹ The exact result reads [28, 29]

$$\begin{aligned} A_{++}(p, p') &= S_{++}^{\alpha\alpha}(p, p') , & B_{++}(p, p') &= S_{++}^{\alpha\alpha}(p, p') \frac{x'^+ - x^+}{x'^+ - x^-} \frac{1}{\nu} , \\ C_{++}(p, p') &= S_{++}^{\alpha\alpha}(p, p') \frac{x'^+ - x'^-}{x'^+ - x^-} \frac{\eta}{\eta'} \sqrt{\frac{\nu'}{\nu}} , & D_{++}(p, p') &= S_{++}^{\alpha\alpha}(p, p') \frac{x'^- - x^-}{x'^+ - x^-} \nu' , \\ E_{++}(p, p') &= S_{++}^{\alpha\alpha}(p, p') \frac{x^+ - x^-}{x'^+ - x^-} \frac{\eta'}{\eta} \sqrt{\frac{\nu'}{\nu}} , & F_{++}(p, p') &= S_{++}^{\alpha\alpha}(p, p') \frac{x'^- - x^+}{x'^+ - x^-} \frac{\nu'}{\nu} , \end{aligned} \quad (3.10)$$

$$\begin{aligned} A_{+-}(p, p') &= S_{+-}^{\alpha\alpha}(p, p') \frac{1 - \frac{1}{x^+ x'^-}}{1 - \frac{1}{x^- x'^-}} \nu , & B_{+-}(p, p') &= -S_{+-}^{\alpha\alpha}(p, p') \frac{i \eta \eta'}{x^- x'^-} \frac{1}{1 - \frac{1}{x^- x'^-}} (\nu \nu')^{-\frac{1}{2}} , \\ C_{+-}(p, p') &= S_{+-}^{\alpha\alpha}(p, p') , & D_{+-}(p, p') &= S_{+-}^{\alpha\alpha}(p, p') \frac{1 - \frac{1}{x^+ x'^+}}{1 - \frac{1}{x^- x'^-}} \nu \nu' , \\ E_{+-}(p, p') &= S_{+-}^{\alpha\alpha}(p, p') \frac{1 - \frac{1}{x^- x'^+}}{1 - \frac{1}{x^- x'^-}} \nu' , & F_{+-}(p, p') &= -S_{+-}^{\alpha\alpha}(p, p') \frac{i \eta \eta'}{x^+ x'^+} \frac{1}{1 - \frac{1}{x^- x'^-}} (\nu \nu')^{\frac{3}{2}} . \end{aligned} \quad (3.11)$$

The structure of the S-matrix is identical to (3.5) and (3.6), the only differences being the overall phase factors, $S_{++}^{\alpha\alpha}(p, p')$ and $S_{+-}^{\alpha\alpha}(p, p')$, and that in the definition of the variables x^\pm given in appendix A.1 the mass should be set to α . The phase factors $S_{\pm\pm}^{\alpha\alpha}$ and $S_{\pm\mp}^{\alpha\alpha}$ have been computed semiclassically in [36].

Various components of the tree-level S-matrix were computed directly in [23, 24]. These are consistent with the near-BMN expansion of the exact result (3.10), (3.11). As in the $AdS_3 \times S^3 \times T^4$ case, the remaining components of the tree-level S-matrix can then be fixed from the expansion of the exact result. We shall

⁸For particles of mass $\bar{\alpha}$ the corresponding result can be obtained simply by replacing α with $\bar{\alpha}$.

⁹To be precise we use the definitions (3.4) with the replacements $\varphi \rightarrow \varphi_2$ and $\psi \rightarrow \chi^2$ and likewise for their conjugates.

present the result in the gauge $a = \frac{1}{2}$ as the dependence on a again goes through the unitarity procedure without any particular subtlety, i.e. it exponentiates as in eq. (3.9). The tree-level S-matrix reads

$$\begin{aligned} A_{++}^{(0)}(p, p') &= \frac{\alpha(p+p')^2}{2(e'p - ep')} , & B_{++}^{(0)}(p, p') &= \frac{\alpha p'(p+p')}{2(e'p - ep')} , \\ C_{++}^{(0)}(p, p') &= pp' \frac{\sqrt{(e+p)(e'-p')} + \sqrt{(e-p)(e'+p')}}{2(e'p - ep')} , & D_{++}^{(0)}(p, p') &= \frac{\alpha p(p+p')}{2(e'p - ep')} , \\ E_{++}^{(0)}(p, p') &= pp' \frac{\sqrt{(e+p)(e'-p')} + \sqrt{(e-p)(e'+p')}}{2(e'p - ep')} , & F_{++}^{(0)}(p, p') &= 0 , \end{aligned} \quad (3.12)$$

$$\begin{aligned} A_{+-}^{(0)}(p, p') &= \frac{\alpha(p-p')^2}{2(e'p - ep')} , & B_{+-}^{(0)}(p, p') &= pp' \frac{\sqrt{(e-p)(e'+p')} - \sqrt{(e+p)(e'-p')}}{2(e'p - ep')} , \\ C_{+-}^{(0)}(p, p') &= \frac{\alpha p'(p'-p)}{2(e'p - ep')} , & D_{+-}^{(0)}(p, p') &= \frac{\alpha p(p-p')}{2(e'p - ep')} , \\ E_{+-}^{(0)}(p, p') &= 0 , & F_{+-}^{(0)}(p, p') &= pp' \frac{\sqrt{(e-p)(e'+p')} - \sqrt{(e+p)(e'-p')}}{2(e'p - ep')} . \end{aligned} \quad (3.13)$$

Let us now turn our attention to the scattering between a mode with mass α and one with mass $\bar{\alpha} = 1 - \alpha$. There are no surprises regarding the gauge-fixing parameter a , i.e. eq. (3.9) also holds for the two mass scattering. We again define the parametrizing functions as

$$\begin{aligned} S_{\varphi_2 \varphi_3}^{\varphi_2 \varphi_3}(p, p') &= A_{++}(p, p') & S_{\varphi_2 \varphi_3}^{\varphi_2 \bar{\varphi}_3}(p, p') &= A_{+-}(p, p') \\ S_{\varphi_2 \chi^3}^{\varphi_2 \chi^3}(p, p') &= B_{++}(p, p') & S_{\varphi_2 \varphi_3}^{\chi^2 \chi^3}(p, p') &= B_{+-}(p, p') \\ S_{\varphi_2 \chi^3}^{\chi^2 \varphi_3}(p, p') &= C_{++}(p, p') & S_{\varphi_2 \chi^3}^{\varphi_2 \chi^3}(p, p') &= C_{+-}(p, p') \\ S_{\chi^2 \varphi_3}^{\chi^2 \varphi_3}(p, p') &= D_{++}(p, p') & S_{\chi^2 \varphi_3}^{\chi^2 \bar{\varphi}_3}(p, p') &= D_{+-}(p, p') \\ S_{\chi^2 \varphi_3}^{\varphi_2 \chi^3}(p, p') &= E_{++}(p, p') & S_{\chi^2 \bar{\chi}^3}^{\chi^2 \chi^3}(p, p') &= E_{+-}(p, p') \\ S_{\chi^2 \chi^3}^{\chi^2 \chi^3}(p, p') &= F_{++}(p, p') & S_{\chi^2 \chi^3}^{\varphi_2 \bar{\varphi}_3}(p, p') &= F_{+-}(p, p') \end{aligned} \quad (3.14)$$

with the functions in string frame given by [28, 29]

$$\begin{aligned} A_{++}(p, p') &= S_{++}^{\alpha \bar{\alpha}}(p, p') , & B_{++}(p, p') &= S_{++}^{\alpha \bar{\alpha}}(p, p') \frac{y'^+ - x^+}{y'^+ - x^-} \frac{1}{\nu} , \\ C_{++}(p, p') &= S_{++}^{\alpha \bar{\alpha}}(p, p') \frac{y'^+ - y'^-}{y'^+ - x^-} \frac{\eta}{\eta'} \sqrt{\frac{\nu'}{\nu}} , & D_{++}(p, p') &= S_{++}^{\alpha \bar{\alpha}}(p, p') \frac{y'^- - x^-}{y'^+ - x^-} \nu' , \\ E_{++}(p, p') &= S_{++}^{\alpha \bar{\alpha}}(p, p') \frac{x^+ - x^-}{y'^+ - x^-} \frac{\eta'}{\eta} \sqrt{\frac{\nu'}{\nu}} , & F_{++}(p, p') &= S_{++}^{\alpha \bar{\alpha}}(p, p') \frac{y'^- - x^+}{y'^+ - x^-} \frac{\nu'}{\nu} , \end{aligned} \quad (3.15)$$

$$\begin{aligned} A_{+-}(p, p') &= S_{+-}^{\alpha \bar{\alpha}}(p, p') \frac{1 - \frac{1}{x^+ y'^-}}{1 - \frac{1}{x^- y'^-}} \nu , & B_{+-}(p, p') &= -S_{+-}^{\alpha \bar{\alpha}}(p, p') \frac{i \eta \eta'}{x^- y'^-} \frac{1}{1 - \frac{1}{x^- y'^-}} (\nu \nu')^{-\frac{1}{2}} , \\ C_{+-}(p, p') &= S_{+-}^{\alpha \bar{\alpha}}(p, p') , & D_{+-}(p, p') &= S_{+-}^{\alpha \bar{\alpha}}(p, p') \frac{1 - \frac{1}{x^+ y'^+}}{1 - \frac{1}{x^- y'^-}} \nu \nu' , \\ E_{+-}(p, p') &= S_{+-}^{\alpha \bar{\alpha}}(p, p') \frac{1 - \frac{1}{x^- y'^+}}{1 - \frac{1}{x^- y'^-}} \nu' , & F_{+-}(p, p') &= -S_{+-}^{\alpha \bar{\alpha}}(p, p') \frac{i \eta \eta'}{x^+ y'^+} \frac{1}{1 - \frac{1}{x^- y'^-}} (\nu \nu')^{\frac{3}{2}} . \end{aligned} \quad (3.16)$$

Here we have defined the overall phase factors by setting

$$\hat{S}_{\varphi_2 \varphi_3}^{\varphi_2 \varphi_3}(p, p') = 1 , \quad \hat{S}_{\varphi_2 \chi^3}^{\varphi_2 \bar{\varphi}_3}(p, p') = 1 . \quad (3.17)$$

The phase factors $S_{\pm\pm}^{\alpha\bar{\alpha}}$ and $S_{\pm\mp}^{\alpha\bar{\alpha}}$ have been computed semiclassically at one loop in [36].

As before, the tree-level S-matrix can be extracted from the near-BMN expansion of the exact result along with those amplitudes computed in [23, 24]. For $a = \frac{1}{2}$ (again the contribution of the gauge-fixing parameter a to the unitarity computation goes through without any particular subtlety) it is given by

$$\begin{aligned} A_{++}^{(0)}(p, p') &= 0, & B_{++}^{(0)}(p, p') &= -\frac{p(\bar{\alpha}p + \alpha p')}{2(e'p - ep')}, \\ C_{++}^{(0)}(p, p') &= pp' \frac{\sqrt{(e+p)(e'-p')} + \sqrt{(e-p)(e'+p')}}{2(e'p - ep')}, & D_{++}^{(0)}(p, p') &= -\frac{p'(\bar{\alpha}p + \alpha p')}{2(e'p - ep')}, \\ E_{++}^{(0)}(p, p') &= pp' \frac{\sqrt{(e+p)(e'-p')} + \sqrt{(e-p)(e'+p')}}{2(e'p - ep')}, & F_{++}^{(0)}(p, p') &= -\frac{(p+p')(\bar{\alpha}p + \alpha p')}{2(e'p - ep')}, \end{aligned} \quad (3.18)$$

$$\begin{aligned} A_{+-}^{(0)}(p, p') &= 0, & B_{+-}^{(0)}(p, p') &= pp' \frac{\sqrt{(e-p)(e'+p')} - \sqrt{(e+p)(e'-p')}}{2(e'p - ep')}, \\ C_{+-}^{(0)}(p, p') &= -\frac{p(\bar{\alpha}p - \alpha p')}{2(e'p - ep')}, & D_{+-}^{(0)}(p, p') &= \frac{p'(\bar{\alpha}p - \alpha p')}{2(e'p - ep')}, \\ E_{+-}^{(0)}(p, p') &= -\frac{(p-p')(\bar{\alpha}p - \alpha p')}{2(e'p - ep')}, & F_{+-}^{(0)}(p, p') &= pp' \frac{\sqrt{(e-p)(e'+p')} - \sqrt{(e+p)(e'-p')}}{2(e'p - ep')}. \end{aligned} \quad (3.19)$$

3.1.3 A general tree-level S-matrix for the $AdS_3 \times S^3 \times M^4$ theories

Comparing the expressions (3.7), (3.8), (3.12), (3.13), (3.18) and (3.19) we notice their similarity. In particular, they all differ from one another by a term proportional to the identity. Therefore in this section we will introduce an additional parameter β along with two generic masses m and m' , such that, for particular values of these three parameters the tree-level S-matrices are recovered. The advantage of this approach is that it demonstrates how some quantities in the one-loop result are common to all three theories (i.e. β -independent) up to the right assignment of the masses.

To be concrete the expression for the general tree-level S-matrix is (we use the notation $\bar{\beta} = (1 - \beta)$)

$$\begin{aligned} A_{++}^{(0)}(p, p') &= \beta \frac{(p+p')(m'p + mp')}{2(e'p - ep')}, & B_{++}^{(0)}(p, p') &= \frac{(\beta p' - \bar{\beta}p)(m'p + mp')}{2(e'p - ep')}, \\ C_{++}^{(0)}(p, p') &= pp' \frac{\sqrt{(e+p)(e'-p')} + \sqrt{(e-p)(e'+p')}}{2(e'p - ep')}, & D_{++}^{(0)}(p, p') &= \frac{(\beta p - \bar{\beta}p')(m'p + mp')}{2(e'p - ep')}, \\ E_{++}^{(0)}(p, p') &= pp' \frac{\sqrt{(e+p)(e'-p')} + \sqrt{(e-p)(e'+p')}}{2(e'p - ep')}, & F_{++}^{(0)}(p, p') &= -\bar{\beta} \frac{(p+p')(m'p + mp')}{2(e'p - ep')}, \\ A_{+-}^{(0)}(p, p') &= \beta \frac{(p-p')(m'p - mp')}{2(e'p - ep')}, & B_{+-}^{(0)}(p, p') &= pp' \frac{\sqrt{(e-p)(e'+p')} - \sqrt{(e+p)(e'-p')}}{2(e'p - ep')}, \\ C_{+-}^{(0)}(p, p') &= \frac{(\bar{\beta}p + \beta p')(m'p - mp')}{2(e'p - ep')}, & D_{+-}^{(0)}(p, p') &= \frac{(\bar{\beta}p' + \beta p)(m'p - mp')}{2(e'p - ep')}, \\ E_{+-}^{(0)}(p, p') &= -\bar{\beta} \frac{(p-p')(m'p - mp')}{2(e'p - ep')}, & F_{+-}^{(0)}(p, p') &= pp' \frac{\sqrt{(e-p)(e'+p')} - \sqrt{(e+p)(e'-p')}}{2(e'p - ep')}. \end{aligned} \quad (3.20)$$

The explicit assignments that need to be made to recover the various tree-level S-matrices given in the previous section are shown in table 2. For most of the unitarity computation however, we will keep general values of β , m and m' so as to better understand the dependence of the result on these parameters.

(β, m, m')	Theory
$(0, \alpha, \bar{\alpha})$	$AdS_3 \times S^3 \times S^3 \times S^1$ (two mass scattering)
$(\frac{1}{2}, 1, 1)$	$AdS_3 \times S^3 \times T^4$
$(1, \alpha, \alpha)$	$AdS_3 \times S^3 \times S^3 \times S^1$ (one mass scattering)

Table 2: Assignments of parameters for the various theories of interest.

3.2 Result from unitarity techniques

In this section we compute the one-loop S-matrix from unitarity methods for the light-cone gauge-fixed string theories in the $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ backgrounds supported by RR flux. As explained in section 2.4, we will split the result according to eqs. (2.45), (2.46) and (2.47), where we recall that we have chosen $S_{\varphi\varphi}^{\varphi\varphi} = A_{++}(p, p')$ and $S_{\varphi\bar{\psi}}^{\varphi\bar{\psi}} = C_{+-}(p, p')$ as the overall phase factors.

In the general construction described in section 2.2, we found that when scattering a particle of mass m with one of mass m' , the s- and u-channel contributions are just given in terms of the tree-level S-matrices for the same scattering configuration. Therefore, as the logarithmic terms (2.45) and the rational terms (2.46) only come from the s-channel and u-channel contributions, for these we can work with the general (β -dependent) tree-level S-matrix (3.20). For the t-channel contribution (2.47) one needs to combine different tree-level S-matrices, for example the scattering of two particles of mass m with the scattering of a particle of mass m with one of mass m' . Hence for these terms we will need to restrict to the specific values of β , m and m' given in table 2.

3.2.1 Coefficients of the logarithms

We start by briefly reviewing the work of [2]. As discussed in section 2.4 one should always be able to include the logarithmic terms of the S-matrix in the phases. Therefore at one loop we expect them to only contribute to the diagonal terms. In [2] the authors proved that this is indeed the case and that furthermore, the particular combination governing the logarithmic dependence does not depend on the diagonal components of the tree-level S-matrix. Therefore, the one-loop logarithmic terms following from the unitarity construction for the general tree-level S-matrix (3.20) will be β -independent. Indeed,

$$\ell_{++}(p, p') = -\frac{p^2 p'^2}{4\pi(ee' - pp' - mm')} , \quad (3.21)$$

$$\ell_{+-}(p, p') = -\frac{p^2 p'^2}{4\pi(ee' - pp' + mm')} , \quad (3.22)$$

where the functions $\ell_{\sigma_M \sigma_N}$ were introduced in eq. (2.42). Although not transparent from this expression, these functions can be expressed as

$$\ell_{++}(p, p') = -\frac{1}{2\pi} C_{++}^{(0)}(p, p') E_{++}^{(0)}(p, p') , \quad (3.23)$$

$$\ell_{+-}(p, p') = -\frac{1}{2\pi} B_{+-}^{(0)}(p, p') F_{+-}^{(0)}(p, p') . \quad (3.24)$$

As pointed out in [2] the fact that these functions can be expressed in terms of the entries of the tree-level S-matrix is clear from the unitarity construction. In the next section we will show how this property extends to the rational part following from the s-channel.

3.2.2 Rational terms from the s-channel

In section 2.4 we described how the contributions to the rational part of the S-matrix in the unitarity calculation are split between the s-channel (2.46) and t-channel (2.47). Let us start by considering the s-channel, for which we can work with the general β -dependent tree-level S-matrix (3.20). From eq. (2.46) it is clear that we can restrict our attention to $\text{Im}(\hat{T}^{(1)})$, where we recall that $\hat{T}^{(0)}$ and $\hat{T}^{(1)}$ are the tree-level and one-loop terms in the expansion of the S-matrix with the overall phase factors, $S_{\varphi\varphi}^{\varphi\varphi} = A_{++}(p, p')$ and $S_{\varphi\psi}^{\varphi\psi} = C_{+-}(p, p')$, set to one. The result from the unitarity calculation is (2.46)

$$\frac{1}{2}\hat{T}^{(0)} \circledast \hat{T}^{(0)} . \quad (3.25)$$

Below we give the components of (3.25). These are in perfect agreement with the one-loop expansion of the exact results (3.5), (3.6), (3.10), (3.11), (3.15) and (3.16) for the appropriate assignments of the masses m and m' , see table 2. The one-loop expressions are

$$\begin{aligned} \hat{A}_{++}^{(1)}(p, p') &= 0 , \\ \hat{B}_{++}^{(1)}(p, p') &= \frac{1}{2} \left[\frac{p(m'p + mp')}{2(e'p - ep')} \right]^2 + \frac{1}{2} \left[pp' \frac{\sqrt{(e+p)(e'-p')} + \sqrt{(e-p)(e'+p')}}{2(e'p - ep')} \right]^2 , \\ \hat{C}_{++}^{(1)}(p, p') &= -\frac{1}{2} \left[\frac{(p+p')(m'p + mp')}{2(e'p - ep')} \right] \left[pp' \frac{\sqrt{(e+p)(e'-p')} + \sqrt{(e-p)(e'+p')}}{2(e'p - ep')} \right] , \\ \hat{D}_{++}^{(1)}(p, p') &= \frac{1}{2} \left[\frac{p'(m'p + mp')}{2(e'p - ep')} \right]^2 + \frac{1}{2} \left[pp' \frac{\sqrt{(e+p)(e'-p')} + \sqrt{(e-p)(e'+p')}}{2(e'p - ep')} \right]^2 , \\ \hat{E}_{++}^{(1)}(p, p') &= -\frac{1}{2} \left[\frac{(p+p')(m'p + mp')}{2(e'p - ep')} \right] \left[pp' \frac{\sqrt{(e+p)(e'-p')} + \sqrt{(e-p)(e'+p')}}{2(e'p - ep')} \right] , \\ \hat{F}_{++}^{(1)}(p, p') &= \frac{1}{2} \left[\frac{(p+p')(m'p + mp')}{2(e'p - ep')} \right]^2 . \quad (3.26) \\ \hat{A}_{+-}^{(1)}(p, p') &= \frac{1}{2} \left[\frac{p(m'p - mp')}{2(e'p - ep')} \right]^2 + \frac{1}{2} \left[pp' \frac{\sqrt{(e+p)(e'-p')} - \sqrt{(e-p)(e'+p')}}{2(e'p - ep')} \right]^2 , \\ \hat{B}_{+-}^{(1)}(p, p') &= -\frac{1}{2} \left[\frac{(p+p')(m'p - mp')}{2(e'p - ep')} \right] \left[pp' \frac{\sqrt{(e+p)(e'-p')} - \sqrt{(e-p)(e'+p')}}{2(e'p - ep')} \right] , \\ \hat{C}_{+-}^{(1)}(p, p') &= 0 , \\ \hat{D}_{+-}^{(1)}(p, p') &= \frac{1}{2} \left[\frac{(p+p')(m'p - mp')}{2(e'p - ep')} \right]^2 , \\ \hat{E}_{+-}^{(1)}(p, p') &= \frac{1}{2} \left[\frac{p'(m'p - mp')}{2(e'p - ep')} \right]^2 + \frac{1}{2} \left[pp' \frac{\sqrt{(e+p)(e'-p')} - \sqrt{(e-p)(e'+p')}}{2(e'p - ep')} \right]^2 , \\ \hat{F}_{+-}^{(1)}(p, p') &= -\frac{1}{2} \left[\frac{(p+p')(m'p - mp')}{2(e'p - ep')} \right] \left[pp' \frac{\sqrt{(e+p)(e'-p')} - \sqrt{(e-p)(e'+p')}}{2(e'p - ep')} \right] . \quad (3.27) \end{aligned}$$

Although there are simpler ways to express this result, we have chosen this form in order to explicitly show the connection with the tree-level functions. The β -independence of (3.26) and (3.27) is expected since β appears only in the phases. As explained earlier in this section and in section 2.4, to check the s-channel rational terms we do not need to consider the overall phase factors and hence they have been set to one.

Note that expressions for the components of $\frac{1}{2}T^{(0)} \circledast T^{(0)}$ in terms of tree-level functions are given in [2] for $AdS_3 \times S^3 \times T^4$. These formulae also hold for the general tree-level S-matrix (3.20), however, they will depend on β , which drops out only if we consider $\frac{1}{2}\hat{T}^{(0)} \circledast \hat{T}^{(0)}$ as above. To see explicitly how this works

let us consider F_{++} .¹⁰ From [2] the one-loop expression for F_{++} is simply given by

$$F_{++}^{(1)} = \frac{1}{2}[F_{++}^{(0)}]^2, \quad (3.28)$$

however when we consider (3.25) (taking into account that $\phi_{++}^{(0)} = A_{++}^{(0)}$) we find

$$\hat{F}_{++}^{(1)} = \frac{1}{2}[\hat{F}_{++}^{(0)}]^2 = \frac{1}{2}[F_{++}^{(0)} - A_{++}^{(0)}]^2. \quad (3.29)$$

Comparing the expressions for $F_{++}^{(0)}$ and $A_{++}^{(0)}$ we can then observe the cancellation of β . A similar story holds for the other components

$$\begin{aligned} \hat{A}_{++}^{(1)} &= 0, & \hat{B}_{++}^{(1)} &= \frac{1}{2}[B_{++}^{(0)} - A_{++}^{(0)}]^2 + \frac{1}{2}C_{++}^{(0)}E_{++}^{(0)}, \\ \hat{C}_{++}^{(1)} &= \frac{1}{2}[B_{++}^{(0)} + D_{++}^{(0)} - 2A_{++}^{(0)}]C_{++}^{(0)}, & \hat{D}_{++}^{(1)} &= \frac{1}{2}[D_{++}^{(0)} - A_{++}^{(0)}]^2 + \frac{1}{2}C_{++}^{(0)}E_{++}^{(0)}, \\ \hat{E}_{++}^{(1)} &= \frac{1}{2}[B_{++}^{(0)} + D_{++}^{(0)} - 2A_{++}^{(0)}]E_{++}^{(0)}, & \hat{F}_{++}^{(1)} &= \frac{1}{2}[F_{++}^{(0)} - A_{++}^{(0)}]^2. \end{aligned} \quad (3.30)$$

$$\begin{aligned} \hat{A}_{+-}^{(1)} &= \frac{1}{2}[A_{+-}^{(0)} - C_{+-}^{(0)}]^2 + \frac{1}{2}B_{+-}^{(0)}F_{+-}^{(0)}, & \hat{B}_{+-}^{(1)} &= \frac{1}{2}[A_{+-}^{(0)} + E_{+-}^{(0)} - 2C_{+-}^{(0)}]B_{+-}^{(0)}, \\ \hat{C}_{+-}^{(1)} &= 0, & \hat{D}_{+-}^{(1)} &= \frac{1}{2}[D_{+-}^{(0)} - C_{+-}^{(0)}]^2, \\ \hat{E}_{+-}^{(1)} &= \frac{1}{2}[E_{+-}^{(0)} - C_{+-}^{(0)}]^2 + \frac{1}{2}B_{+-}^{(0)}F_{+-}^{(0)}, & \hat{F}_{+-}^{(1)} &= \frac{1}{2}[A_{+-}^{(0)} + E_{+-}^{(0)} - 2C_{+-}^{(0)}]F_{+-}^{(0)}. \end{aligned} \quad (3.31)$$

The validity of these relations is rather general and can be applied to any S-matrix with the same underlying structure. In particular, this allows us to use them for the mixed flux case in section 4.

3.2.3 The t-channel contribution and the dressing phases

As explained in section 2 the t-channel cut requires a non-trivial generalization of the procedure used for the $AdS_5 \times S^5$ case [1]. Furthermore, the t-channel cut for the scattering of two masses depends on the tree-level S-matrices for the scattering of the same and different masses. Therefore, in this section it only makes sense to work with the parameters β , m and m' for the three cases of interest, as given in table 2. Inputting the tree-level S-matrices (3.7), (3.8), (3.12), (3.13), (3.18) and (3.19) into eq. (2.28) and splitting the result as in eq. (2.47) we find for all three scattering processes ($AdS_3 \times S^3 \times T^4$, $AdS_3 \times S^3 \times S^3 \times S^1$ same mass and $AdS_3 \times S^3 \times S^3 \times S^1$ different mass) the one-loop phases can be written in the following general form

$$\phi_{++}^{(1)}(p, p') = \frac{p p' (m' p + m p')^2}{8\pi m m' (e' p - e p')}, \quad (3.32)$$

$$\phi_{+-}^{(1)}(p, p') = -\frac{p p' (m' p - m p')^2}{8\pi m m' (e' p - e p')}. \quad (3.33)$$

The real part of the one-loop cut-constructible S-matrix that is not part of the overall phase factors is given by

$$\text{Re}(\hat{T}^{(1)})|_{\text{unit.}} = \frac{1}{4\pi} |1 - 2\beta| \left(\frac{p^2}{m} + \frac{p'^2}{m'} \right) T^{(0)}. \quad (3.34)$$

It is important to emphasise that even though we have written them in terms of β , m and m' the results (3.32), (3.33) and (3.34) are only valid for the assignments in table 2.

¹⁰For the remainder of this section the dependence on p and p' is understood.

Two comments are in order here. First, eq. (3.34) is proportional to $|1 - 2\beta|$. Therefore, this term vanishes for $AdS_3 \times S^3 \times T^4$, but does not for $AdS_3 \times S^3 \times S^3 \times S^1$. However, we should recall that this is only the contribution to $\text{Re}(\hat{T}^{(1)})$ coming from unitarity and there are potentially additional terms arising from external leg corrections (2.47). Indeed, one of the main differences between $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ is that the light-cone gauge-fixed Lagrangian of the latter has cubic terms. Furthermore, the tree-level form factor for one off-shell and two on-shell particles is non-zero and as a consequence non-trivial external leg corrections are already present at one loop in the unitarity construction, as described in section 2.3. As we will see in the following section these precisely cancel (3.34) and re-establish agreement with the exact result.^{11,12}

The second comment concerns eqs. (3.21), (3.22), (3.32) and (3.33), which combined have a natural interpretation as the one-loop contributions to the phases. It is interesting to note that they are independent of β , indicating that the phases for all three scattering processes should be related. This agrees with the semiclassical computation [15].¹³ A natural question is whether this relation extends to all orders in the coupling. Although this goes beyond the scope of this work, to facilitate comparison with the literature [31] we will rewrite the result in terms of the standard strong coupling variables x and y , which we have defined in (A.6) and (A.7)

$$\varpi_{++}^{(1)}(p, p') = -\frac{mm'}{\pi} \frac{x^2}{x^2-1} \frac{y^2}{y^2-1} \left[\frac{(x+y)^2(1-\frac{1}{xy})}{(x^2-1)(x-y)(y^2-1)} + \frac{2}{(x-y)^2} \log \left(\frac{x+1}{x-1} \frac{y-1}{y+1} \right) \right], \quad (3.35)$$

$$\varpi_{+-}^{(1)}(p, p') = -\frac{mm'}{\pi} \frac{x^2}{x^2-1} \frac{y^2}{y^2-1} \left[\frac{(xy+1)^2(\frac{1}{x}-\frac{1}{y})}{(x^2-1)(xy-1)(y^2-1)} + \frac{2}{(xy-1)^2} \log \left(\frac{x+1}{x-1} \frac{y-1}{y+1} \right) \right]. \quad (3.36)$$

Here x corresponds to momentum p with mass m and y to momentum p' with mass m' . Finally, let us stress again that this expression is valid for all three cases summarized in table 2. In particular, for $m = m' = 1$ this is consistent with (B.12), where the overall sign is compensated by the fact that $e^{i\vartheta_{\sigma_M \sigma_N}(p, p')} \sim S_{\sigma_M \sigma_N}^{11}(p, p')^{-1}$, see eqs. (B.1) and (B.2).

3.2.4 External leg corrections for $AdS_3 \times S^3 \times S^3 \times S^1$

In this section we focus on the $AdS_3 \times S^3 \times S^3 \times S^1$ background for which the unwanted term (3.34) is present. With the aim of interpreting this missing term as a contribution cancelled by external leg corrections let us review the results of [27, 26] for the one-loop two-point functions. The near-BMN expansion of the light-cone gauge-fixed Lagrangian can be schematically written as

$$\mathcal{L} = \mathcal{L}_2 + h^{-\frac{1}{2}} \mathcal{L}_3 + h^{-1} \mathcal{L}_4 + \dots \quad (3.37)$$

The quadratic part is given by¹⁴

$$\mathcal{L}_2 = \bar{\chi}^a (i\not{\partial} - m_a) \chi^a + |\partial \varphi_a|^2 - m_a^2 |\varphi_a|^2, \quad (3.38)$$

¹¹Let us point out that a term like (3.34) in the one-loop S-matrix would prevent the latter from satisfying the Yang-Baxter equation, conflicting with the integrability of the theory.

¹²It is interesting to note that in the two loop near-flat-space computation of [40] for the $AdS_5 \times S^5$ light-cone gauge S-matrix the external leg corrections also cancelled unwanted terms arising from t-channel graphs.

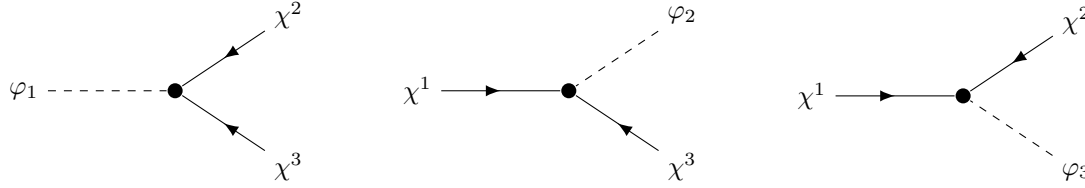
¹³In [36] the author states that the one-loop dressing phase of $AdS_3 \times S^3 \times S^3 \times S^1$ is half that of $AdS_3 \times S^3 \times T^4$. This is consistent given that we are considering the factorized S-matrix for $AdS_3 \times S^3 \times T^4$.

¹⁴Here we stress that, although the theory is not Lorentz invariant beyond quadratic order, we are formally rearranging the fermions into doublets for notational and computational convenience.

where our conventions are summarized in appendix A and we have introduced the index $a = 1, \dots, 4$ with the respective masses listed in table 1. The cubic Lagrangian [27, 26] is given by

$$\begin{aligned} \mathcal{L}_3 = \sqrt{\frac{\alpha\bar{\alpha}}{2}} & \left[(\chi^1)^T \gamma^3 (i\not{\partial} - \alpha) \varphi_2 \chi^3 - i(\chi^1)^T \gamma^3 (i\not{\partial} - \bar{\alpha}) \varphi_3 \chi^2 - 2(\chi^2)^T \gamma^1 \partial_1 \varphi_1 \chi^3 \right. \\ & + \bar{\chi}^2 \gamma^0 (i\not{\partial} - \alpha) \varphi_2 \chi^4 + i\bar{\chi}^3 \gamma^0 (i\not{\partial} - \bar{\alpha}) \varphi_3 \chi^4 \\ & \left. - (\bar{\chi}^2 (1 - \gamma^3) \chi^2 - \bar{\chi}^3 (1 - \gamma^3) \chi^3 + 2\alpha |\varphi_2|^2 - 2\bar{\alpha} |\varphi_3|^2) \partial_0 \varphi_4 + \text{h.c.} \right]. \end{aligned} \quad (3.39)$$

Let us start by focusing on the tree-level processes following from the cubic Lagrangian. The only processes allowed by two-dimensional kinematics involve a particle of mass 1 decaying into a particle of mass α and one of mass $\bar{\alpha}$ and its reverse.¹⁵ The Feynman rules associated to the relevant vertices are



$$\begin{aligned} & \sqrt{\frac{\alpha\bar{\alpha}}{2}} 2i\gamma^1 p_1, & -\sqrt{\frac{\alpha\bar{\alpha}}{2}} i\gamma^3 (\not{p}_2 + \alpha), & -\sqrt{\frac{\alpha\bar{\alpha}}{2}} \gamma^3 (\not{p}_3 + \bar{\alpha}). \end{aligned} \quad (3.40)$$

To obtain the amplitude one should contract the external legs with the fermion polarizations and enforce the on-shell condition. The three diagrams share the same on-shell kinematics, i.e. denoting the incoming momentum of the heavy particle (with mass $m_1 = 1$) as p_1 , the outgoing momenta of the light particles are given by $p_2 = \frac{m_2}{m_1} p_1$ and $p_3 = \frac{m_3}{m_1} p_1$, where $m_3 = m_1 - m_2$.¹⁶ Using the property that $v(m_i p_1) = \sqrt{m_i} v(p_1)$ (see eq. (A.12)), it is clear that both the second and the third diagrams vanish as $(\not{p} + 1)v(p) = 0$. Furthermore, the first diagram is also identically zero as a consequence of the identity $v(p)^T \gamma^1 v(p) = 0$.

One may ask how this is compatible with the result of [27] where the authors find a non-vanishing expression for the one-loop correction to the propagators coming from the graph formed of two three-point vertices. Focusing on the one-loop contribution to the self-energy of the heavy boson the result of [27] reads

$$\Sigma_0^{(1)}(p) = i \langle \varphi_1 \bar{\varphi}_1 \rangle^{(1)} = \frac{1}{\pi^2} (\alpha \log \alpha + \bar{\alpha} \log \bar{\alpha}) p^2. \quad (3.41)$$

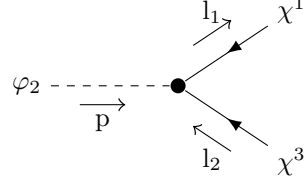
This result is obtained setting $p^2 = 1$ (i.e. putting the propagator on-shell) and its dependence on p is a consequence of the lack of Lorentz invariance. In a unitarity computation with the setup described in section 2.3 the two tree-level form factors appearing in figure 3 would be vanishing in the strict on-shell limit and this contribution would not be caught. However, as discussed in section 2.3, our treatment ignored any kind of tadpole diagram contributing to the external leg corrections. Moreover, as pointed out in [27] the contribution (3.41) can be understood as the one-loop term in the expansion of $h(h)$. Combining this observation with the fact that, in all the examples we have considered, ignoring tadpole diagrams gives the S-matrix up to corrections in $h(h)$ we may argue that these are coming from tadpole diagrams whose analysis would require the introduction of a regularization (see also [27]). This is therefore an additional indication that unitarity techniques, neglecting tadpoles, are blind to shifts in the coupling.

¹⁵Diagrams involving one massless leg are ruled out by two-dimensional kinematics. In the cubic Lagrangian (3.39) the massless modes always couple to massive modes of equal mass. It then follows that the on-shell condition implies that the massless leg carries vanishing momentum.

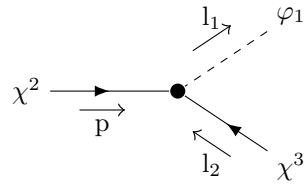
¹⁶This is true under the assumption of a relativistic dispersion relation, which in this case holds just at tree level.

Therefore, we will consider the following alternative question. Are there are external leg corrections that are caught by unitarity and which are relevant for the one-loop calculation? In the S-matrix computation we consider scattering processes for which the external legs have masses α or $\bar{\alpha}$. Therefore, the external leg corrections we compute come from diagrams similar to the first graph in figure 2 with masses $m_1 = 1$ and $m_2 = \alpha$ or $m_2 = \bar{\alpha}$.

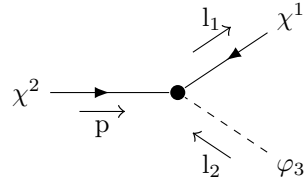
We start by considering an external leg of mass α . Using the vertices in eq. (3.40) we find the following form factors



$$= i \sqrt{\frac{\alpha\bar{\alpha}}{2}} v(l_1)^T \gamma^3 (\not{p} - \alpha) u(l_2) , \quad (3.42)$$



$$= i \sqrt{\frac{\alpha\bar{\alpha}}{2}} 2 u(l_2)^T \gamma^1 l_1 u(p) , \quad (3.43)$$



$$= \sqrt{\frac{\alpha\bar{\alpha}}{2}} v(l_1)^T \gamma^3 (\not{l}_2 - \bar{\alpha}) u(p) . \quad (3.44)$$

In order to apply the construction outlined in section 2.3 we need to compute eq. (2.39). In particular, we are interested in expanding the form factor squared around the on-shell condition. Since we already know that the tree-level form factor vanishes on-shell, to get the first order in the expansion there is no need to also expand the integral, i.e. it can be evaluated strictly on-shell

$$I(\alpha^2, 1, \bar{\alpha}) = -\frac{i}{4\pi\bar{\alpha}} . \quad (3.45)$$

Squaring the form factor (3.42) and expanding around the on-shell condition we find¹⁷

$$-i\Sigma_{1,\varphi_2}^{(1)}(p) = \frac{i}{4\pi\alpha} p^2 . \quad (3.46)$$

Comparing to (3.34) this result is promising. However, (3.46) holds only when the external leg is a boson. A non-trivial check of our procedure is that when the external leg is a fermion the correction, which comes from two terms associated to the diagrams (3.43) and (3.44), is exactly the same as for the boson, i.e.

$$-i\Sigma_{1,\chi^2}^{(1)}(p) = \frac{i}{4\pi\alpha} p^2 . \quad (3.47)$$

One might have expected this from worldsheet supersymmetry as discussed in [26]. Here we have computed the external leg corrections for a particle of mass α . From the symmetry of the Lagrangian, it is clear that the result for a particle of mass $\bar{\alpha}$ is just given by the replacement $\alpha \rightarrow \bar{\alpha}$.

Once the external leg contributions are computed we can apply eq. (2.35) to find their contribution to the one-loop S-matrix. To be general, let us consider the scattering of a particle of mass m with a particle

¹⁷A minus sign is included to take account of the fermion loop.

of mass m' . Our result then reads

$$T_{ext}^{(1)} = -\frac{1}{4\pi} \left(\frac{p^2}{m} + \frac{p'^2}{m'} \right) T^{(0)} . \quad (3.48)$$

This contribution exactly cancels (3.34) for $\beta = 0$ and $\beta = 1$. These are precisely the values associated to the single and mixed mass scattering processes for $AdS_3 \times S^3 \times S^3 \times S^1$, and hence we have established agreement between the unitarity calculation and the exact result up to shifts in the coupling.

4 Massive sector for $AdS_3 \times S^3 \times T^4$ supported by mixed flux

In this section we apply the methods of section 2 to the massive sector of the $AdS_3 \times S^3 \times T^4$ light-cone gauge-fixed string theory supported by a mix of RR and NSNS fluxes. We parameterize the relative strength of the two fluxes with a parameter q , with $q = 0$ corresponding to pure RR and $q = 1$ to pure NSNS flux. The symmetry algebras underlying this theory and its S-matrix are the same for all q [17, 21, 25, 32] – the parameter q enters through the form of the representation of the algebra on the states.

4.1 Exact S-matrix and tree-level result

The quadratic light-cone gauge-fixed action for the $AdS_3 \times S^3 \times T^4$ background supported by mixed flux again describes 4 + 4 massive and 4 + 4 massless fields. As usual we restrict ourselves to considering the scattering of two massive excitations to two massive excitations. Following the RR case described in section 3.1.1 we group the particle content of the massive sector into 2 + 2 complex degrees of freedom (to recall, $\Phi_{\varphi\varphi}$, $\Phi_{\psi\psi}$, $\Phi_{\varphi\psi}$, $\Phi_{\psi\varphi}$, and their complex conjugates $\Phi_{\bar{\varphi}\bar{\varphi}}$, $\Phi_{\bar{\psi}\bar{\psi}}$, $\Phi_{\bar{\varphi}\bar{\psi}}$, $\Phi_{\bar{\psi}\bar{\varphi}}$). The presence of the NSNS flux then breaks the charge conjugation invariance, such that the near-BMN dispersion relations for these complex degrees of freedom are given by

$$e_{\pm} = \sqrt{(1 - q^2) + (p \pm q)^2} . \quad (4.1)$$

where $+$ corresponds to $\Phi_{\varphi\varphi}$, $\Phi_{\psi\psi}$, $\Phi_{\varphi\psi}$, $\Phi_{\psi\varphi}$ and $-$ to their complex conjugates.

As for $q = 0$ the S-matrix factorizes as in (3.1) and the general structure of the factorized S-matrix takes the form given in (2.40) with $\sigma_{\varphi} = \sigma_{\psi} = +$ and $\sigma_{\bar{\varphi}} = \sigma_{\bar{\psi}} = -$. Furthermore, the construction outlined in section 2 still gives the same one-loop result whether we consider the factorized or full S-matrix. Therefore, for simplicity we will again work with the former. Due to the lack of charge conjugation symmetry all four phases are now different. However, charge conjugation along with formally sending $q \rightarrow -q$ is a symmetry and hence $\phi_{++} = \phi_{--}|_{q \rightarrow -q}$ and $\phi_{+-} = \phi_{-+}|_{q \rightarrow -q}$. Similarly, for the functions $\ell_{\sigma_M \sigma_N}$ we have $\ell_{++} = \ell_{--}|_{q \rightarrow -q}$ and $\ell_{+-} = \ell_{-+}|_{q \rightarrow -q}$. Therefore, in the following we will again focus on the $++$ and $+-$ sectors. The dependence on the gauge-fixing parameter a is also modified in the following natural way

$$\exp \left[\frac{i}{2} \left(a - \frac{1}{2} \right) (\epsilon'_{\sigma_N} p - \epsilon_{\sigma_M} p') \right] , \quad (4.2)$$

where the all-order energies ϵ_{\pm} are defined in appendix A.2. As discussed in section 2.4 we choose the overall phase factors by setting particular components of \hat{S}_{MN}^{PQ} to one

$$\hat{S}_{\varphi\varphi}^{\varphi\varphi}(p, p') = 1 , \quad \hat{S}_{\varphi\psi}^{\varphi\bar{\psi}}(p, p') = 1 . \quad (4.3)$$

The parametrizing functions of the exact S-matrix are defined as

$$\begin{aligned}
S_{\varphi\varphi}^{\varphi\varphi}(p, p') &= A_{++}(p, p') & S_{\varphi\bar{\varphi}}^{\varphi\bar{\varphi}}(p, p') &= A_{+-}(p, p') \\
S_{\varphi\psi}^{\varphi\psi}(p, p') &= B_{++}(p, p') & S_{\varphi\bar{\varphi}}^{\psi\bar{\psi}}(p, p') &= B_{+-}(p, p') \\
S_{\varphi\psi}^{\psi\varphi}(p, p') &= C_{++}(p, p') & S_{\varphi\bar{\varphi}}^{\psi\bar{\varphi}}(p, p') &= C_{+-}(p, p') \\
S_{\psi\varphi}^{\psi\varphi}(p, p') &= D_{++}(p, p') & S_{\psi\bar{\varphi}}^{\psi\bar{\varphi}}(p, p') &= D_{+-}(p, p') \\
S_{\psi\varphi}^{\psi\psi}(p, p') &= E_{++}(p, p') & S_{\psi\bar{\varphi}}^{\psi\bar{\psi}}(p, p') &= E_{+-}(p, p') \\
S_{\psi\psi}^{\psi\psi}(p, p') &= F_{++}(p, p') & S_{\psi\bar{\varphi}}^{\varphi\bar{\varphi}}(p, p') &= F_{+-}(p, p')
\end{aligned} \tag{4.4}$$

with the functions in string frame given by [32]

$$\begin{aligned}
A_{++}(p, p') &= S_{++}(p, p') , & B_{++}(p, p') &= S_{++}(p, p') \frac{x_+^+ - x_+^-}{x_+^+ - x_+^-} \frac{1}{\nu_+} , \\
C_{++}(p, p') &= S_{++}(p, p') \frac{x_+^+ - x_+^-}{x_+^+ - x_+^-} \frac{\eta_+}{\eta_+} \sqrt{\frac{\nu_+}{\nu_+}} , & D_{++}(p, p') &= S_{++}(p, p') \frac{x_+^+ - x_+^-}{x_+^+ - x_+^-} \nu_+ , \\
E_{++}(p, p') &= S_{++}(p, p') \frac{x_+^+ - x_+^-}{x_+^+ - x_+^-} \frac{\eta_+}{\eta_+} \sqrt{\frac{\nu_+}{\nu_+}} , & F_{++}(p, p') &= S_{++}(p, p') \frac{x_+^+ - x_+^-}{x_+^+ - x_+^-} \frac{\nu_+}{\nu_+} ,
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
A_{+-}(p, p') &= S_{+-}(p, p') \frac{1 - \frac{1}{x_+^+ x_+^-}}{1 - \frac{1}{x_+^+ x_+^-}} \nu_+ , & B_{+-}(p, p') &= -S_{+-}(p, p') \frac{i \eta_+ \eta_-}{x_+^+ x_+^-} \frac{1}{1 - \frac{1}{x_+^+ x_+^-}} (\nu_+ \nu_-)^{-\frac{1}{2}} , \\
C_{+-}(p, p') &= S_{+-}(p, p') , & D_{+-}(p, p') &= S_{+-}(p, p') \frac{1 - \frac{1}{x_+^+ x_+^-}}{1 - \frac{1}{x_+^+ x_+^-}} \nu_+ \nu_- , \\
E_{+-}(p, p') &= S_{+-}(p, p') \frac{1 - \frac{1}{x_+^+ x_+^-}}{1 - \frac{1}{x_+^+ x_+^-}} \nu_- , & F_{+-}(p, p') &= -S_{+-}(p, p') \frac{i \eta_+ \eta_-}{x_+^+ x_+^-} \frac{1}{1 - \frac{1}{x_+^+ x_+^-}} (\nu_+ \nu_-)^{\frac{3}{2}} .
\end{aligned} \tag{4.6}$$

The definitions of the variables entering these expressions are given in appendix A.2. The functions $S_{++}(p, p')$ and $S_{+-}(p, p')$ are two of the overall phase factors. These phase factors are not fixed by symmetry – they are constrained by crossing symmetry, however, they are currently unknown.

The input needed for the unitarity construction of section 2 is the tree-level S-matrix. Various tree-level components were computed directly in [25]. These are in agreement with the near-BMN expansion of the exact result (4.5), (4.6), for which we recall that the integrable coupling used in the definition of x_{\pm}^{\pm} and x_{\pm}^{\pm} has the expansion $h(h) = h + \mathcal{O}(h^0)$ and to take the near-BMN limit the spatial momenta should be rescaled $p \rightarrow \frac{p}{h}$. The remaining components of the tree-level S-matrix can then be fixed from the expansion of the exact result. As in the RR case, here we shall present the result in the gauge $a = \frac{1}{2}$ – the dependence on a goes through the unitarity procedure without any particular subtlety, i.e. it exponentiates as in eq. (4.2). The tree-level S-matrix reads

$$\begin{aligned}
A_{++}^{(0)}(p, p') &= -F_{++}^{(0)}(p, p') = \frac{(p + p')(e'_+ p + e_+ p')}{4(p - p')} , \\
C_{++}^{(0)}(p, p') &= E_{++}^{(0)}(p, p') = \frac{p p'}{2(p - p')} \left(\sqrt{(e_+ + p + q)(e'_+ + p' + q)} + \sqrt{(e_+ - p - q)(e'_+ - p' - q)} \right) , \\
B_{++}^{(0)}(p, p') &= -D_{++}^{(0)}(p, p') = -\frac{e'_+ p - e_+ p'}{4} ,
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
A_{+-}^{(0)}(p, p') &= -E_{+-}^{(0)}(p, p') = \frac{(p - p')(e'_- p + e_+ p')}{4(p + p')} , \\
B_{+-}^{(0)}(p, p') &= F_{+-}^{(0)}(p, p') = \frac{pp'}{2(p + p')} \left(\sqrt{(e_+ - p - q)(e'_- - p' + q)} - \sqrt{(e_+ + p + q)(e'_- + p' - q)} \right) , \\
C_{+-}^{(0)}(p, p') &= -D_{+-}^{(0)}(p, p') = -\frac{e'_- p - e_+ p'}{4} ,
\end{aligned} \tag{4.8}$$

This form of writing the tree-level S-matrix elements is the simplest for the purposes of introducing the parameter q . Agreement with (3.7) and (3.8) for $q = 0$ can be checked using the dispersion relation.

4.2 Result from unitarity techniques

In this section we compute the one-loop S-matrix from unitarity methods for the light-cone gauge-fixed string theory in the $AdS_3 \times S^3 \times T^4$ background supported by a mix of RR and NSNS fluxes. Again, we will split the result according to eqs. (2.45), (2.46) and (2.47), where we recall that we have chosen $S_{\varphi\varphi}^{\varphi\varphi} = A_{++}(p, p')$ and $S_{\varphi\psi}^{\varphi\bar{\psi}} = C_{+-}(p, p')$ as the overall phase factors.

There is a subtlety regarding the unitarity computation in that the near-BMN dispersion relations (4.1) are not the standard relativistic ones that we assumed for the derivation in section 2. To bypass this problem, we will first shift the momenta as

$$p \rightarrow p - q \text{ for particles and } p \rightarrow p + q \text{ for antiparticles} , \tag{4.9}$$

so as to put the near-BMN dispersion relations into the standard form. At the level of the light-cone gauge-fixed Lagrangian this just amounts to a σ -dependent rotation of the complex fields, where σ is the spatial coordinate on the worldsheet [25]. We can then straightforwardly use the construction of section 2 for two particles of mass $\sqrt{1 - q^2}$. To construct the one-loop result, we should then conclude by undoing the shift (4.9). An analogous approach was used in [2] to compute to logarithmic terms.

Following this procedure it is apparent that the logarithms appearing in the one-loop integrals, when written in terms of energy and momentum, are different for each of the four sectors

$$\theta_{\pm\pm} = \text{arcsinh} \left(\frac{e'_\pm(p \pm q) - e_\pm(p' \pm q)}{1 - q^2} \right) , \quad \theta_{\pm\mp} = \text{arcsinh} \left(\frac{e'_\mp(p \pm q) - e_\pm(p' \mp q)}{1 - q^2} \right) . \tag{4.10}$$

The functions $\ell_{\sigma_M \sigma_N}$ are then defined as the coefficients of $\theta_{\sigma_M \sigma_N}$ in the one-loop phase, see eq. (2.42).

We start by briefly reviewing the work of [2]. Given that the structure of the S-matrix is not altered by the presence of NSNS flux it follows from the unitarity computation that the coefficients of the logarithms written in terms of the tree-level functions, (4.7) and (4.8), are still given by (3.23) and (3.24)

$$\ell_{++}(p, p') = -\frac{1}{2\pi} C_{++}^{(0)}(p, p') E_{++}^{(0)}(p, p') = -\frac{p^2 p'^2 (e_+ e'_+ + (p + q)(p' + q) + (1 - q^2))}{4\pi(p - p')^2} , \tag{4.11}$$

$$\ell_{+-}(p, p') = -\frac{1}{2\pi} B_{+-}^{(0)}(p, p') F_{+-}^{(0)}(p, p') = -\frac{p^2 p'^2 (e_+ e'_- + (p + q)(p' - q) - (1 - q^2))}{4\pi(p + p')^2} . \tag{4.12}$$

Using the dispersion relation, one can check that these expressions agree with eqs. (3.21) and (3.22) for $q = 0$ and $m = m' = 1$.

Furthermore, the rational s-channel terms (with the overall phase factors set to one) are again given in terms of the tree-level functions as in eqs. (3.30) and (3.31). Plugging in the corresponding expressions, (4.7) and (4.8), one can check agreement with the near-BMN expansion of the exact result (4.5) and (4.6).

Finally, as for the $AdS_3 \times S^3 \times T^4$ background supported by pure RR flux, the rational contributions from the t-channel go completely into the phases. That is $\text{Re}(\hat{T}^{(1)})|_{\text{unit.}} = 0$. Furthermore, also as for the case of pure RR flux, the light-cone gauge-fixed Lagrangian contains no cubic terms. Therefore, there are correspondingly no external leg corrections at one loop in the unitarity computation. It follows from computing the t-channel cuts that

$$\phi_{++}^{(1)}(p, p') = \frac{p p' (p + p') (e'_+ p + e_+ p')}{8\pi(p - p')}, \quad (4.13)$$

$$\phi_{+-}^{(1)}(p, p') = -\frac{p p' (p - p') (e'_- p + e_+ p')}{8\pi(p + p')}. \quad (4.14)$$

Using the dispersion relation, one can check that these expressions agree with eqs. (3.32) and (3.33) for $q = 0$ and $m = m' = 1$.

We conclude this section by giving the generalization of the one-loop dressing phases (3.35) and (3.36) in the presence of NSNS flux. As discussed in appendix A.2 the standard strong coupling variables x and y are modified for $q \neq 0$. In particular, we now have a separate variable for the particle x_+ , y_+ and the antiparticle x_- , y_- . These are defined in (A.16) and (A.17). Our conjecture for the one-loop dressing phases is then given by (x_{\pm} corresponds to p and y_{\pm} to p')

$$\begin{aligned} \varpi_{++}^{(1)}(p, p') = & -\frac{1}{\pi} \frac{x_+^2}{\sqrt{1-q^2}(x_+^2-1)-2qx_+} \frac{y_+^2}{\sqrt{1-q^2}(y_+^2-1)-2qy_+} \\ & \left[\frac{(x_+ + y_+)(\sqrt{1-q^2}(x_+ + y_+)(1 - \frac{1}{x_+ y_+}) - 4q)}{(\sqrt{1-q^2}(x_+^2-1)-2qx_+)(x_+ - y_+)(\sqrt{1-q^2}(y_+^2-1)-2qy_+)} \right. \\ & \left. + \frac{2}{(x_+ - y_+)^2} \log \left(\frac{\sqrt{1+q} x_+ + \sqrt{1-q}}{\sqrt{1-q} x_+ - \sqrt{1+q}} \frac{\sqrt{1-q} y_+ - \sqrt{1+q}}{\sqrt{1+q} y_+ + \sqrt{1-q}} \right) \right], \end{aligned} \quad (4.15)$$

$$\begin{aligned} \varpi_{+-}^{(1)}(p, p') = & -\frac{1}{\pi} \frac{x_+^2}{\sqrt{1-q^2}(x_+^2-1)-2qx_+} \frac{y_-^2}{\sqrt{1-q^2}(y_-^2-1)+2qy_-} \\ & \left[\frac{(x_+ y_- + 1)(\sqrt{1-q^2}(x_+ y_- + 1)(\frac{1}{x_+} - \frac{1}{y_-}) + 4q)}{(\sqrt{1-q^2}(x_+^2-1)-2qx_+)(x_+ - y_-)(\sqrt{1-q^2}(y_-^2-1)+2qy_-)} \right. \\ & \left. + \frac{2}{(x_+ y_- - 1)^2} \log \left(\frac{\sqrt{1+q} x_+ + \sqrt{1-q}}{\sqrt{1-q} x_+ - \sqrt{1+q}} \frac{\sqrt{1+q} y_- - \sqrt{1-q}}{\sqrt{1-q} y_- + \sqrt{1+q}} \right) \right]. \end{aligned} \quad (4.16)$$

5 Comments

In this paper we have applied one-loop unitarity methods to the massive sectors of three $AdS_3 \times S^3 \times M^4$ light-cone gauge-fixed string theories. To do so we extended the construction of [1] to account for both particles of different mass in the asymptotic spectrum and external leg corrections. Applying the results to the $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ backgrounds supported by RR flux correctly reproduces the expansion of the exact S-matrix [30, 31] and matches semiclassical computations [34, 35, 36] (see footnote 2). This agreement crucially included the rational terms, supporting the conjecture of [1] that the S-matrices of integrable field theories are cut-constructible (up to a possible shift in the coupling).

The final theory we investigated was $AdS_3 \times S^3 \times T^4$ supported by a mix of RR and NSNS fluxes. For this theory the one-loop unitarity computation reproduced the exact result up to phases, which in this case

are not known beyond tree level. Under the assumption that the unitarity computation still reproduces all the rational terms, we gave a conjecture for the one-loop dressing phases.

All three of these theories contain massless modes in their spectrum. Recently an exact form for the S-matrix describing the scattering of massless modes in the $AdS_3 \times S^3 \times T^4$ background was conjectured in [41], and various perturbative computations have been reported in [23]. It would therefore be of clear interest to extend the unitarity methods to include these excitations.

Massless modes are of interest also for scattering on top of the GKP string [42, 43], for which excitations on the sphere governed by the $O(6)$ non-linear sigma model [44]. This S-matrix recently turned out to be useful for studying $\mathcal{N} = 4$ scattering amplitudes in the collinear limit [45, 46].

There are a number of further possible applications of the methods described in this work (along with those in [1, 2]). These include the study of other integrable string backgrounds [20, 47] and more general off-shell objects, including form factors [48, 49] and correlation functions. Finally, one of the most important directions for future study is the derivation of the rational terms from unitarity methods beyond one loop. This would require a deeper understanding of both how to treat tadpole diagrams and shifts in the coupling or lack thereof.

Note added: While this paper was in the final stages of preparation the article [50] was announced on arXiv. Among other things, the authors reported on the semiclassical computation of the one-loop phases for the $AdS_3 \times S^3 \times T^4$ mixed flux background. The conjecture for these phases given in this paper agrees with the result of [50].

Acknowledgments

We would like to thank V. Forini for fruitful discussions and enjoyable collaborations on related work. We are also grateful to B. Basso, R. Borsato, O. Engelund, S. Komatsu, M. Meineri, J. Plefka, R. Roiban, A. Tseytlin and P. Vieira for useful discussions and to R. Roiban and A. Tseytlin for valuable comments on the draft. This work is funded by DFG via the Emmy Noether Program ‘‘Gauge Fields from Strings.’’. LB would like to thank Perimeter Institute for Theoretical Physics for the kind hospitality during the preparation of this work.

A Notation and conventions

A.1 $AdS_3 \times S^3 \times M^4$ supported by RR flux

In section 3.1 the exact S-matrices are written as functions of the Zhukovsky variables x^\pm and y^\pm . These are defined in terms of the energy and momentum as follows

$$\frac{x^+}{x^-} = e^{ip}, \quad x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-} = \frac{2i\epsilon}{h}, \quad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2im}{h}, \quad (\text{A.1})$$

$$\frac{y^+}{y^-} = e^{ip}, \quad y^+ - \frac{1}{y^+} - y^- + \frac{1}{y^-} = \frac{2i\epsilon}{h}, \quad y^+ + \frac{1}{y^+} - y^- - \frac{1}{y^-} = \frac{2im'}{h}, \quad (\text{A.2})$$

where h is the integrable coupling that is (potentially non-trivially) related to the string tension h . The third equation of each line is a constraint that is interpreted as the dispersion relation. In particular, m

and m' are the masses of the respective particles. The variables x'^{\pm} and y'^{\pm} are simply given by sending $p \rightarrow p'$ and $\epsilon \rightarrow \epsilon'$. Solving for x^{\pm} and y^{\pm} in terms of p we find

$$x^{\pm} = \frac{e^{\pm i \frac{p}{2}} (m + \epsilon)}{2 h \sin \frac{p}{2}}, \quad \epsilon = \sqrt{m^2 + 4 h^2 \sin^2 \frac{p}{2}}, \quad (\text{A.3})$$

$$y^{\pm} = \frac{e^{\pm i \frac{p}{2}} (m' + \epsilon)}{2 h \sin \frac{p}{2}}, \quad \epsilon = \sqrt{m'^2 + 4 h^2 \sin^2 \frac{p}{2}}. \quad (\text{A.4})$$

When expanding in near-BMN regime, the spatial momenta should first be rescaled as $p \rightarrow \frac{p}{h}$ where h is the string tension. The integrable coupling h , in principle, is related to h in a non-trivial way, however, its strong coupling expansion starts with $h(h) = h + \mathcal{O}(h^0)$. Therefore, at leading order in the near-BMN expansion the dispersion relation is given by its relativistic counterpart, e . The two additional functions that we use to write the expressions for the exact S-matrices are

$$\eta = \sqrt{i(x^- - x^+)} , \quad \nu = \sqrt{\frac{x^+}{x^-}} , \quad (\text{A.5})$$

and similarly for y^{\pm} when referring to a particle of mass m' .

In section 3.2.3 we are interested in expanding the functions x^{\pm} and y^{\pm} at strong coupling. To do so it is convenient to introduce a new variable x such that

$$x^{\pm} = x \pm \frac{im}{h} \frac{x^2}{x^2 - 1} + \mathcal{O}(h^{-3}) . \quad (\text{A.6})$$

Expressing x in terms of p in the near-BMN expansion (i.e. first rescaling p) one finds

$$x(p) = \frac{m + \sqrt{m^2 + p^2}}{p} + \mathcal{O}(h^{-2}) . \quad (\text{A.7})$$

Using the new variable one can easily expand the dressing phase at strong coupling as shown in appendix B.

In the discussion of the $AdS_3 \times S^3 \times S^1$ background we need to use the cubic terms in the expansion of the light-cone gauge-fixed Lagrangian. We use a worldsheet metric with signature $(+, -)$. Light-cone coordinates are defined for a generic two-dimensional vector v^{μ} as $v^{\pm} = \frac{1}{2}(v^0 \pm v^1)$ and for a covector v_{μ} as $v_{\pm} = v_0 \pm v_1$. The non-vanishing elements of the metric in light-cone coordinates are $\eta_{+-} = \eta_{-+} = 2$. Correspondingly $\eta^{+-} = \eta^{-+} = \frac{1}{2}$. The Levi-Civita tensor is defined as $\epsilon^{01} = 1 = -\epsilon_{01}$.

As usual, gamma matrices are defined by the anti-commutation relation

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2 \eta^{\mu\nu} . \quad (\text{A.8})$$

An explicit representation is given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^3 = -\gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.9})$$

A generic spinor is represented as

$$\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}, \quad (\text{A.10})$$

where χ_{\pm} are the chiral projections of χ by the projectors $P_{\pm} = \frac{1}{2}(1 \pm \gamma^3)$. The conjugation is defined in the usual way $\bar{\chi} = \chi^{\dagger} \gamma^0$ and to make contact with [27, 26] we define $\bar{\chi}_{\pm} \equiv \chi_{\pm}^{\dagger}$. The polarization vectors can be chosen to be purely real and given by

$$\text{---}\blacktriangleright\text{---}\bigcirc\text{---} \quad u(p) = \begin{pmatrix} \sqrt{p_+} \\ \sqrt{p_-} \end{pmatrix}, \quad (\text{A.11})$$

$$\text{Diagram: A circle filled with dots, with an arrow pointing to the right.} \quad v(\mathbf{p}) = \begin{pmatrix} \sqrt{p_+} \\ -\sqrt{p_-} \end{pmatrix}. \quad (\text{A.12})$$

A.2 $AdS_3 \times S^3 \times T^4$ supported by mixed flux

In the mixed flux case discussed in section 4 the S-matrix is again written in terms of Zhukovsky-type variables. However, the dispersion relation is modified and is different for particles (x_+^\pm) and antiparticles (x_-^\pm). The Zhukovsky variables are defined in terms of the energy and momentum as follows

$$\frac{x_{\pm}^+}{x_{\pm}^-} = e^{ip}, \quad x_{\pm}^+ - \frac{1}{x_{\pm}^+} - x_{\pm}^- + \frac{1}{x_{\pm}^-} = \frac{2i\epsilon_{\pm}}{\hbar\sqrt{1-q^2}}. \quad (\text{A.13})$$

However, the dispersion relation [33] is now given by

$$\sqrt{1-q^2}\left(x_{\pm}^+ + \frac{1}{x_{\pm}^+} - x_{\pm}^- - \frac{1}{x_{\pm}^-}\right) \mp 2q \log \frac{x_{\pm}^+}{x_{\pm}^-} = \frac{2i}{h}, \quad (\text{A.14})$$

The variables x'_\pm and x_\pm are simply given by sending $p \rightarrow p'$ and $\epsilon_\pm \rightarrow \epsilon'_\pm$. Solving for x'_\pm and x_\pm in terms of p we find

$$\begin{aligned} x_{\pm}^{\pm} &= \frac{e^{\pm i \frac{\pi}{2}} (1 + q p + \epsilon_{\pm}(p))}{2 h \sqrt{1 - q^2} \sin \frac{p}{2}}, & x_{\pm}^{\pm} &= \frac{e^{\pm i \frac{\pi}{2}} (1 - q p + \epsilon_{\mp}(p))}{2 h \sqrt{1 - q^2} \sin \frac{p}{2}}, \\ \epsilon_{\pm} &= \sqrt{(1 \pm q h p)^2 + 4 h^2 (1 - q^2) \sin^2 \frac{p}{2}}. \end{aligned} \quad (\text{A.15})$$

As expected, at leading order in the near-BMN expansion the dispersion relation is given by e_{\pm} as defined in (4.1). The functions η_{\pm} and ν_{\pm} are generalized in the obvious way from (A.5).

In section 4.2 we are interested in expanding the functions x_{\pm}^{\pm} at strong coupling. To do so it is convenient to introduce new variables x_{\pm} such that

$$\begin{aligned} x_{+}^{\pm} &= x_{+} \pm \frac{i}{h} \frac{x_{+}^2}{\sqrt{1-q^2}(x_{+}^2-1)-2qh x_{+}} + \mathcal{O}(h^{-3}) , \\ x_{-}^{\pm} &= x_{-} \pm \frac{i}{h} \frac{x_{-}^2}{\sqrt{1-q^2}(x_{-}^2-1)+2qh x_{-}} + \mathcal{O}(h^{-3}) . \end{aligned} \quad (\text{A.16})$$

Expressing x_{\pm} in terms of p in the near-BMN expansion (i.e. first rescaling p) one finds

$$x_{\pm}(p) = \frac{1 \pm qp + \sqrt{(1 \pm qp)^2 + (1 - q^2)p^2}}{\sqrt{1 - q^2}p} + \mathcal{O}(h^{-2}) . \quad (\text{A.17})$$

B Phase factors for $AdS_3 \times S^3 \times M^4$ backgrounds

In this appendix we give the relevant details regarding the dressing phases for the RR backgrounds and their expansion in the near-BMN limit.

In eqs. (3.5) and (3.6) $S_{++}^{11}(p, p')$ and $S_{+-}^{11}(p, p')$ appeared as overall phase factors in the exact result. In the case of $AdS_3 \times S^3 \times T^4$ their expressions are known exactly [31]

$$S_{++}^{11}(p, p')^{-1} = e^{-\frac{i}{2}a(\epsilon'p - \epsilon p')} \sqrt{\frac{x'^- - x^+}{x'^+ - x^-} \frac{1 - \frac{1}{x^+x'^-}}{1 - \frac{1}{x^-x'^+}}} \frac{\nu'}{\nu} e^{i\vartheta_{++}^{11}(x^\pm, x'^\pm)}, \quad (B.1)$$

$$S_{+-}^{11}(p, p')^{-1} = e^{-\frac{i}{2}a(\epsilon'p - \epsilon p')} \sqrt{\frac{1 - \frac{1}{x^+x'^+}}{1 - \frac{1}{x^-x'^-}} \frac{1 - \frac{1}{x^+x'^-}}{1 - \frac{1}{x^-x'^+}}} \nu' e^{i\vartheta_{+-}^{11}(x^\pm, x'^\pm)}. \quad (B.2)$$

The functions $\vartheta_{++}^{11}(p, p')$ and $\vartheta_{+-}^{11}(p, p')$ can be expressed in terms of an auxiliary function χ

$$\vartheta_{++}^{11}(x^\pm, x'^\pm) = \chi(x^+, x'^+) + \chi(x^-, x'^-) - \chi(x^+, x'^-) - \chi(x^-, x'^+), \quad (B.3)$$

$$\vartheta_{+-}^{11}(x^\pm, x'^\pm) = \tilde{\chi}(x^+, x'^+) + \tilde{\chi}(x^-, x'^-) - \tilde{\chi}(x^+, x'^-) - \tilde{\chi}(x^-, x'^+), \quad (B.4)$$

and the explicit all-order expressions for χ and $\tilde{\chi}$ are

$$\chi(x, y) = \chi^{\text{BES}}(x, y) + \frac{1}{2}(-\chi^{\text{HL}}(x, y) + \chi^-(x, y)), \quad (B.5)$$

$$\tilde{\chi}(x, y) = \chi^{\text{BES}}(x, y) + \frac{1}{2}(-\chi^{\text{HL}}(x, y) - \chi^-(x, y)). \quad (B.6)$$

Here the function χ^{BES} is the same as that which appears in the $AdS_5 \times S^5$ dressing factor [12], χ^{HL} is the Hernandez Lopez phase [51] and is given by the one-loop term in the strong coupling expansion of χ^{BES} , while the function χ^- does not appear in the $AdS_5 \times S^5$ light-cone gauge S-matrix. The three functions can be expressed compactly as contour integrals

$$\chi^{\text{BES}}(x, y) = i \oint \frac{dw}{2\pi i} \oint \frac{dw'}{2\pi i} \frac{1}{x-w} \frac{1}{y-w'} \log \frac{\Gamma[1 + i\hbar(w + 1/w - w' - 1/w')]}{\Gamma[1 - i\hbar(w + 1/w - w' - 1/w')]}, \quad (B.7)$$

$$\chi^{\text{HL}}(x, y) = \frac{\pi}{2} \oint \frac{dw}{2\pi i} \oint \frac{dw'}{2\pi i} \frac{1}{x-w} \frac{1}{y-w'} \text{sign}(w' + 1/w' - w - 1/w), \quad (B.8)$$

$$\chi^-(x, y) = \oint \frac{dw}{8\pi} \frac{1}{x-w} \log \left[(y-w) \left(1 - \frac{1}{yw} \right) \right] \text{sign}((w - 1/w)/i) - x \leftrightarrow y. \quad (B.9)$$

We are interested in the near-BMN expansion of these expressions. Therefore, let us quote the first two orders of $\vartheta_{++}^{11}(x^\pm, x'^\pm)$ and $\vartheta_{+-}^{11}(x^\pm, x'^\pm)$

$$\vartheta_{++}^{11}(x^\pm, x'^\pm) = \frac{1}{h} \vartheta^{\text{AFS}}(x, x') + \frac{1}{h^2} \vartheta_{++}^{(1)}(x, x') + \mathcal{O}(h^{-3}), \quad (B.10)$$

$$\vartheta_{+-}^{11}(x^\pm, x'^\pm) = \frac{1}{h} \vartheta^{\text{AFS}}(x, x') + \frac{1}{h^2} \vartheta_{+-}^{(1)}(x, x') + \mathcal{O}(h^{-3}), \quad (B.11)$$

The functions appearing in (B.10) and (B.11) are given by

$$\begin{aligned} \vartheta^{\text{AFS}}(x, y) &= \frac{2(x-y)}{(x^2-1)(xy-1)(y^2-1)} + \mathcal{O}(h^{-2}), \\ \vartheta_{++}^{(1)}(x, y) &= \frac{1}{\pi} \frac{x^2}{x^2-1} \frac{y^2}{y^2-1} \left[\frac{(x+y)^2(1-\frac{1}{xy})}{(x^2-1)(x-y)(y^2-1)} + \frac{2}{(x-y)^2} \log \left(\frac{x+1}{x-1} \frac{y-1}{y+1} \right) \right] + \mathcal{O}(h^{-1}), \\ \vartheta_{+-}^{(1)}(x, y) &= \frac{1}{\pi} \frac{x^2}{x^2-1} \frac{y^2}{y^2-1} \left[\frac{(xy+1)^2(\frac{1}{x}-\frac{1}{y})}{(x^2-1)(xy-1)(y^2-1)} + \frac{2}{(xy-1)^2} \log \left(\frac{x+1}{x-1} \frac{y-1}{y+1} \right) \right] + \mathcal{O}(h^{-1}). \end{aligned} \quad (B.12)$$

It is important to point out that the pre-factors appearing in (B.1) and (B.2) can be written as a phase factor whose exponent has a vanishing one-loop ($\mathcal{O}(\hbar^{-2})$) term. This property, together with (B.12), allows us to compare $\vartheta_{++}^{(1)}$ and $\vartheta_{+-}^{(1)}$ directly with our perturbative result following from unitarity methods.

The same property also holds for $AdS_3 \times S^3 \times S^3 \times S^1$, in which case we have a total of four undetermined phase factors

$$S_{++}^{\alpha\alpha}(p, p')^{-1} = e^{-ia(\epsilon' p - \epsilon p')} \frac{1 - \frac{1}{x^+ x'^-}}{1 - \frac{1}{x^- x'^+}} \frac{x'^- - x^+}{x'^+ - x^-} \left(\frac{\nu'}{\nu} \right)^2 e^{i\vartheta_{++}^{\alpha\alpha}(x^\pm, x'^\pm)}, \quad (\text{B.13})$$

$$S_{+-}^{\alpha\alpha}(p, p')^{-1} = e^{-ia(\epsilon' p - \epsilon p')} \sqrt{\frac{1 - \frac{1}{x^+ x'^+}}{1 - \frac{1}{x^- x'^-}} \frac{1 - \frac{1}{x^+ x'^-}}{1 - \frac{1}{x^- x'^+}}} \nu' e^{i\vartheta_{+-}^{\alpha\alpha}(x^\pm, x'^\pm)}, \quad (\text{B.14})$$

and

$$S_{++}^{\alpha\bar{\alpha}}(p, p')^{-1} = e^{-ia(\epsilon' p - \epsilon p')} \frac{1 - \frac{1}{x^+ y'^-}}{1 - \frac{1}{x^- y'^+}} \frac{\nu'}{\nu} e^{i\vartheta_{++}^{\alpha\bar{\alpha}}(x^\pm, x'^\pm)}, \quad (\text{B.15})$$

$$S_{+-}^{\alpha\bar{\alpha}}(p, p')^{-1} = e^{-ia(\epsilon' p - \epsilon p')} \sqrt{\frac{1 - \frac{1}{x^+ y'^+}}{1 - \frac{1}{x^- y'^-}} \left(\frac{1 - \frac{1}{x^+ y'^-}}{1 - \frac{1}{x^- y'^+}} \right)^{\frac{3}{2}}} \nu' e^{i\vartheta_{+-}^{\alpha\bar{\alpha}}(x^\pm, x'^\pm)}. \quad (\text{B.16})$$

Unlike the $AdS_3 \times S^3 \times T^4$ case all-order expressions for $\vartheta_{\sigma_M \sigma_N}^{\alpha\alpha}$ and $\vartheta_{\sigma_M \sigma_N}^{\alpha\bar{\alpha}}$ are not known. The one-loop near-BMN expansions for these phases are given in eqs. (3.35), (3.36) and are essentially the same as (B.12) up to an overall scaling depending on the masses.

References

- [1] L. Bianchi, V. Forini and B. Hoare, “Two-dimensional S -matrices from unitarity cuts”, *JHEP* **1307**, 088 (2013), [arXiv:1304.1798](#).
- [2] O. T. Engelund, R. W. McKeown and R. Roiban, “Generalized unitarity and the worldsheet S matrix in $AdS_n \times S^n \times M^{10-2n}$ ”, *JHEP* **1308**, 023 (2013), [arXiv:1304.4281](#).
- [3] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, “One loop n point gauge theory amplitudes, unitarity and collinear limits”, *Nucl.Phys.* **B425**, 217 (1994), [hep-ph/9403226](#).
- [4] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, “Fusing gauge theory tree amplitudes into loop amplitudes”, *Nucl.Phys.* **B435**, 59 (1995), [hep-ph/9409265](#).
- [5] “Scattering amplitudes in gauge theories: progress and outlook”, edited by R. Roiban, M. Spradlin and A. Volovich, *J.Phys.* **A44**, 450301 (2011).
- [6] A. B. Zamolodchikov and A. B. Zamolodchikov, “Factorized S -Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models”, *Annals Phys.* **120**, 253 (1979).
- [7] P. Dorey, “Exact S matrices”, [hep-th/9810026](#).
- [8] R. Metsaev and A. A. Tseytlin, “Type IIB superstring action in $AdS_5 \times S^5$ background”, *Nucl.Phys.* **B533**, 109 (1998), [hep-th/9805028](#).
- [9] I. Bena, J. Polchinski and R. Roiban, “Hidden symmetries of the $AdS_5 \times S^5$ superstring”, *Phys.Rev.* **D69**, 046002 (2004), [hep-th/0305116](#).
- [10] T. Klose, T. McLoughlin, R. Roiban and K. Zarembo, “Worldsheet scattering in $AdS_5 \times S^5$ ”, *JHEP* **0703**, 094 (2007), [hep-th/0611169](#).
- [11] N. Beisert, “The $SU(2|2)$ dynamic S -matrix”, *Adv.Theor.Math.Phys.* **12**, 945 (2008), [hep-th/0511082](#).

- [12] N. Beisert, B. Eden and M. Staudacher, “*Transcendentality and Crossing*”, *J.Stat.Mech.* **0701**, P01021 (2007), [hep-th/0610251](#).
- [13] G. Arutyunov and S. Frolov, “*Foundations of the $AdS_5 \times S^5$ Superstring. Part I*”, *J.Phys. A* **42**, 254003 (2009), [arXiv:0901.4937](#).
- [14] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond et al., “*Review of AdS/CFT Integrability: An Overview*”, *Lett.Math.Phys.* **99**, 3 (2012), [arXiv:1012.3982](#).
- [15] M. C. Abbott, J. Murugan, P. Sundin and L. Wulff, “*Scattering in AdS_2/CFT_1 and the BES Phase*”, *JHEP* **1310**, 066 (2013), [arXiv:1308.1370](#).
- [16] S. Frolov, J. Plefka and M. Zamaklar, “*The $AdS_5 \times S^5$ superstring in light-cone gauge and its Bethe equations*”, *J.Phys. A* **39**, 13037 (2006), [hep-th/0603008](#).
- [17] I. Pesando, “*The GS type IIB superstring action on $AdS_3 \times S^3 \times T^4$* ”, *JHEP* **9902**, 007 (1999), [hep-th/9809145](#).
- [18] J. Rahmfeld and A. Rajaraman, “*The GS string action on $AdS_3 \times S^3$ with Ramond-Ramond charge*”, *Phys.Rev. D* **60**, 064014 (1999), [hep-th/9809164](#).
- [19] A. Babichenko, B. Stefanski Jr. and K. Zarembo, “*Integrability and the AdS_3/CFT_2 correspondence*”, *JHEP* **1003**, 058 (2010), [arXiv:0912.1723](#).
- [20] K. Zarembo, “*Strings on Semisymmetric Superspaces*”, *JHEP* **1005**, 002 (2010), [arXiv:1003.0465](#).
- [21] A. Cagnazzo and K. Zarembo, “*B-field in AdS_3/CFT_2 Correspondence and Integrability*”, *JHEP* **1211**, 133 (2012), [arXiv:1209.4049](#).
- [22] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, “*Strings in flat space and pp waves from $N=4$ superYang-Mills*”, *JHEP* **0204**, 013 (2002), [hep-th/0202021](#).
- [23] P. Sundin and L. Wulff, “*Worldsheet scattering in AdS_3/CFT_2* ”, *JHEP* **1307**, 007 (2013), [arXiv:1302.5349](#).
- [24] N. Rughoonauth, P. Sundin and L. Wulff, “*Near BMN dynamics of the $AdS_3 \times S^3 \times S^3 \times S^1$ superstring*”, *JHEP* **1207**, 159 (2012), [arXiv:1204.4742](#).
- [25] B. Hoare and A. Tseytlin, “*On string theory on $AdS_3 \times S^3 \times T^4$ with mixed 3-form flux: Tree-level S-matrix*”, *Nucl.Phys. B* **873**, 682 (2013), [arXiv:1303.1037](#).
- [26] P. Sundin, “*Worldsheet two- and four-point functions at one loop in AdS_3/CFT_2* ”, [arXiv:1403.1449](#).
- [27] P. Sundin and L. Wulff, “*Classical integrability and quantum aspects of the $AdS_3 \times S^3 \times S^3 \times S^1$ superstring*”, *JHEP* **1210**, 109 (2012), [arXiv:1207.5531](#).
- [28] R. Borsato, O. Ohlsson Sax and A. Sfondrini, “*A dynamic $\mathfrak{su}(1|1)^2$ S-matrix for AdS_3/CFT_2* ”, *JHEP* **1304**, 113 (2013), [arXiv:1211.5119](#).
- [29] R. Borsato, O. Ohlsson Sax and A. Sfondrini, “*All-loop Bethe ansatz equations for AdS_3/CFT_2* ”, *JHEP* **1304**, 116 (2013), [arXiv:1212.0505](#).
- [30] R. Borsato, O. Ohlsson Sax, A. Sfondrini, B. Stefanski Jr. and A. Torrielli, “*The all-loop integrable spin-chain for strings on $AdS_3 \times S^3 \times T^4$: the massive sector*”, *JHEP* **1308**, 043 (2013), [arXiv:1303.5995](#).
- [31] R. Borsato, O. O. Sax, A. Sfondrini, B. Stefanski Jr. and A. Torrielli, “*Dressing phases of AdS_3/CFT_2* ”, *Phys.Rev. D* **88**, 066004 (2013), [arXiv:1306.2512](#).
- [32] B. Hoare and A. Tseytlin, “*Massive S-matrix of $AdS_3 \times S^3 \times T^4$ superstring theory with mixed 3-form flux*”, *Nucl.Phys. B* **873**, 395 (2013), [arXiv:1304.4099](#).
- [33] B. Hoare, A. Stepanchuk and A. Tseytlin, “*Giant magnon solution and dispersion relation in string theory in $AdS_3 \times S^3 \times T^4$ with mixed flux*”, *Nucl.Phys. B* **879**, 318 (2014), [arXiv:1311.1794](#).
- [34] M. C. Abbott, “*Comment on Strings in $AdS_3 \times S^3 \times S^3 \times S^1$ at One Loop*”, *JHEP* **1302**, 102 (2013), [arXiv:1211.5587](#).

- [35] M. Beccaria, F. Levkovich-Maslyuk, G. Macorini and A. Tseytlin, “Quantum corrections to spinning superstrings in $AdS_3 \times S^3 \times M^4$: determining the dressing phase”, *JHEP* **1304**, 006 (2013), [arXiv:1211.6090](#).
- [36] M. C. Abbott, “The $AdS_3 \times S^3 \times S^3 \times S^1$ Hernandez-Lopez phases: a semiclassical derivation”, *J. Phys. A* **46**, 445401 (2013), [arXiv:1306.5106](#).
- [37] R. Britto and E. Mirabella, “Single Cut Integration”, *JHEP* **1101**, 135 (2011), [arXiv:1011.2344](#).
- [38] O. T. Engelund and R. Roiban, “Correlation functions of local composite operators from generalized unitarity”, *JHEP* **1303**, 172 (2013), [arXiv:1209.0227](#).
- [39] A. Brandhuber, G. Travaglini and G. Yang, “Analytic two-loop form factors in $N=4$ SYM”, *JHEP* **1205**, 082 (2012), [arXiv:1201.4170](#).
- [40] T. Klose, T. McLoughlin, J. Minahan and K. Zarembo, “World-sheet scattering in $AdS_5 \times S^5$ at two loops”, *JHEP* **0708**, 051 (2007), [arXiv:0704.3891](#).
- [41] R. Borsato, O. O. Sax, A. Sfondrini and B. Stefanski Jr., “All-loop worldsheet S matrix for $AdS_3 \times S^3 \times T^4$ ”, [arXiv:1403.4543](#).
- [42] S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory”, *Phys.Lett. B* **428**, 105 (1998), [hep-th/9802109](#).
- [43] B. Basso and A. Rej, “Bethe ansätze for GKP strings”, *Nucl.Phys. B* **879**, 162 (2014), [arXiv:1306.1741](#).
- [44] L. F. Alday and J. M. Maldacena, “Comments on operators with large spin”, *JHEP* **0711**, 019 (2007), [arXiv:0708.0672](#).
- [45] B. Basso, A. Sever and P. Vieira, “Spacetime and Flux Tube S -Matrices at Finite Coupling for $N=4$ Supersymmetric Yang-Mills Theory”, *Phys.Rev.Lett.* **111**, 091602 (2013), [arXiv:1303.1396](#).
- [46] B. Basso, A. Sever and P. Vieira, “On the collinear limit of scattering amplitudes at strong coupling”, [arXiv:1405.6350](#).
- [47] L. Wulff, “Superisometries and integrability of superstrings”, [arXiv:1402.3122](#).
- [48] T. Klose and T. McLoughlin, “Worldsheet Form Factors in AdS/CFT ”, *Phys.Rev. D* **87**, 026004 (2013), [arXiv:1208.2020](#).
- [49] T. Klose and T. McLoughlin, “Comments on World-Sheet Form Factors in AdS/CFT ”, [arXiv:1307.3506](#).
- [50] A. Babichenko, A. Dekel and O. O. Sax, “Finite-gap equations for strings on $AdS_3 \times S^3 \times T^4$ with mixed 3-form flux”, [arXiv:1405.6087](#).
- [51] R. Hernandez and E. Lopez, “Quantum corrections to the string Bethe ansatz”, *JHEP* **0607**, 004 (2006), [hep-th/0603204](#).